GENERALIZED SOLUTIONS TO SEMILINEAR ELLIPTIC PDE WITH APPLICATIONS TO THE LICHNEROWICZ EQUATION

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ABSTRACT. In this article we investigate the existence of a solution to a semi-linear, elliptic, partial differential equation with distributional coefficients and data. The problem we consider is a generalization of the Lichnerowicz equation that one encounters in studying the constraint equations in general relativity. Our method for solving this problem consists of solving a net of regularized, semi-linear problems with data obtained by smoothing the original, distributional coefficients. In order to solve these regularized problems, we develop *a priori* L^{∞}-bounds and sub- and super-solutions to apply a fixed point argument. We then show that the net of solutions obtained through this process satisfies certain decay estimates by determining estimates for the sub- and super-solutions and utilizing classical, *a priori* elliptic estimates. The estimates for this net of solutions allow us to regard this collection of functions as a solution in a Colombeau-type algebra. We motivate this Colombeau algebra framework by first solving an ill-posed critical exponent problem. To solve this ill-posed problem, we use a collection of smooth, "approximating" problems and then use the resulting sequence of solutions and a compactness argument to obtain a solution to the original problem. This approach is modeled after the more general Colombeau framework that we develop, and it conveys the potential that solutions in these abstract spaces have for obtaining classical solutions to ill-posed non-linear problems with irregular data.

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1. INTRODUCTION: SEMILINEAR PROBLEMS AND CRITICAL NONLINEARITIES

In this paper we consider a family of elliptic, semilinear Dirichlet problems that are of the form

$$-\sum_{i,j=1}^{N} D_i(a^{ij}D_ju) + \sum_{i=1}^{K} b^i u^{n_i} = 0 \quad \text{in } \Omega, \tag{1.1}$$

$$u|_{\partial\Omega} = \rho, \tag{1.2}$$

where a^{ij}, b^i and ρ are potentially distributional and $n_i \in \mathbb{Z}$ for each i. These problems are a generalization of the the Lichnerowicz Equation that appears in the study of the constraint equations of General Relativity. The need to understand an equation of this form with rough data arises if one attempts to study equations such as the Lichnerowicz equation when the metric of the embedded hypersurface is not smooth. From a physical perspective, such problems are interesting because distributional metrics correspond to the initial data for physically plausible spacetimes generated by strings and gravitational waves [4, 6]. From a mathematical point of view, developing results for solutions to the Lichnerowicz equation under such low regularity conditions is of interest in that it extends the current "rough metric" existence theory, as described in [8, 10, 9, 2]. The hope is to eventually extend the solution theory to cover rough data examples such as those studied by Maxwell in [11].

The main contributions of this article are an existence result for (1.1) in a Colombeau-type algebra, and an existence result in $W^{1,2}(\Omega)$ for an ill-posed, critical exponent problem of the form

$$-\Delta u + au^5 + bu^i = 0 \text{ in } \Omega, \tag{1.3}$$

$$u|_{\partial\Omega} = \rho,$$

where $1 \leq i \leq 4$ is in $\mathbb{N}$, $\Omega \subset \mathbb{R}^3$, $a \in L^p(\Omega)$ and $b \in L^q$ with $\frac{6}{5} \leq p \leq q$. The framework we use to prove existence for (1.1) consists of embedding the singular data and coefficients into a Colombeau-type algebra so that multiplication of the distributional coefficients is well-defined. To solve (1.3), we do not explicitly require the Colombeau machinery that we develop to solve (1.1), but we use similar ideas to produce a sequence of functions that converge to a solution of (1.3) in $W^{1,2}(\Omega)$.

The Colombeau solution framework for this paper is based mainly on the ideas found in [12]. Here we extend the work done by Mitrovic and Pilipovic in [12] to include a certain collection of semilinear problems. While Pilipovic and Scarpalezos solved a divergent type, quasilinear problem in a Colombeau type algebra in [13], the class of nonlinear problems we consider here does not fit naturally into that framework. Here we provide a solution method that is distinct from those posed in [13] and [12] that is better suited for the class of semilinear problems that we are interested in solving. The set up of our problem is completely similar to the set-up in [12]: given the semilinear Dirichlet problem in (1.1), we consider the family of problems

$$P_\epsilon(x, D)u_\epsilon = f_\epsilon(x, u_\epsilon) \quad \text{on } \Omega, \tag{1.4}$$

$$u_\epsilon|_{\partial\Omega} = \rho_\epsilon,$$

where f_ϵ, h_ϵ, and $P_\epsilon(x, D)$ are obtained by convolving the data and coefficients of (1.1) with a certain mollifier. Thus a solution to the problem in a Colombeau algebra is a net of solutions to the above family satisfying certain decay estimates in ϵ. This is discussed in detail in Sections 3.2 and 3.4. This basic concept underlies both the solution process in our paper and in [12] and [13]. However it is our solution process in the Colombeau

algebra that is quite distinct from that laid out in [12], where the authors used linear elliptic theory to determine a family of solutions and then classical elliptic, *a priori* estimates to prove certain decay estimates. Most notably, the authors developed a precise maximum principle-type argument necessary to obtain the decay estimates required to find a solution. Our strategy for solving (1.1) differs in a number of ways. First, in Section 5.1 we develop a family of *a priori* L^∞ bounds to the family of problems (1.4). Then in Section 5.2 we show that these estimates determine sub- and super-solutions to (1.4). We then employ the method of sub- and super-solutions in Section 4.1 to determine a family of solutions. Finally, ϵ-decay estimates on the sub- and super-solutions are established in Section 5.2, and in Section 6 these estimates are used in conjunction with the *a priori* estimates in Section 3.1 to prove the necessary ϵ-decay estimates on our family of solutions.

This paper can be broken down into two distinct, but related parts. The first part is dedicated to solving (1.3). Our solution to this problem does not explicitly require the techniques that we develop to solve problems with distributional data in Colombeau algebras and only relies on standard elliptic PDE theory. However, the ideas that we use to solve the problem are closely related: we obtain a solution by solving a family of problems similar to (1.4) and then show that these solutions converge to a function in $W^{1,2}(\Omega)$. Therefore, we present our existence result for (1.3) first to convey the benefit that the more general Colombeau solution strategy has, not only for solving problems in the Colombeau Algebra, but also for obtaining solutions in more classical spaces. The remainder of the paper is dedicated to developing the Colombeau framework described in the preceding paragraph. This consists of defining an algebra appropriate for a Dirichlet problem and properly defining a semilinear elliptic problem in the algebra. Once a well-posed elliptic problem in the Colombeau algebra has been formed, we discuss the conditions under which the problem has a solution in the algebra and finally, describe how to translate a given problem of the form (1.1) into a problem that can be solved in the algebra. It should be noted that while the intention is to find solutions to (1.1), the main result pertaining to Colombeau algebras in this paper is Theorem 4.1, which is the main solution result for semilinear problems in our particular Colombeau algebra.

Outline of the paper. The remainder of the paper is structured as follows: In Section 2 we motivate this article by proving the existence of a solution to (1.3). In Section 3 we state a number of preliminary results and develop the technical tools required to solve (1.1). Among these tools and results are the explicit *a priori* estimates found in [12] and a description of the Colombeau framework in which the coefficients and data will be embedded. Then in Section 4 we state the main existence result in Theorem 4.1, give a statement and proof of the method of sub- and super solutions in Theorem 4.3, and then give an outline of the method of proof of Theorem 4.1. Following our discussion of elliptic problems in Colombeau algebras, we discuss a method to embed (1.1) into the algebra to apply our Colombeau existence theory. The remainder of the paper is dedicated to developing the tools to prove Theorem 4.1. In Section 5 we determine *a priori* L^∞ bounds of solutions to our semilinear problem and a net of sub- and super-solutions satisfying explicit ϵ-decay estimates. Finally, in Section 6 we utilize the results from Section 5 to prove the main result outlined in Section 4.

2. Solution Construction using a Sequence of Approximate Problems

If $\Omega \subset \mathbb{R}^3$, the Sobolev embedding theorem tells us that $W^{1,2}(\Omega)$ will compactly embed into $L^p(\Omega)$ for $1 \leq p < 6$ and continuously embed for $1 \leq p \leq 6$. Given functions $u, v \in W^{1,2}(\Omega)$, this upper bound on p places a constraint on the values of i that allow for

the product $u^i v$ to be integrable. In particular, Sobolev embedding and standard Holder inequalities imply that this product will be integrable for arbitrary elements of $W^{1,2}(\Omega)$ only if $1 \leq i \leq 5$. More generally, if $a \in L^\infty(\Omega)$, the term $au^5 v$ will also be integrable. However, if a is an unbounded function in $L^p(\Omega)$ for some $p \geq 1$, then this product is not necessarily integrable without some sort of *a priori* bounds on a, u, and v. Therefore, the following problem does not have a well-defined weak formulation in $W^{1,2}(\Omega)$:

$$-\Delta u + au^5 + bu^i = 0 \quad \text{in } \Omega, \qquad (2.1)$$
$$u|_{\partial\Omega} = \rho,$$

where $1 \leq i \leq 4$ is in $\mathbb{N}$, $\rho \in W^{1,2}(\Omega')$, $a \in L^p(\Omega')$, $b \in L^q(\Omega')$ for $\frac{6}{4} \leq p \leq q$ and where $\Omega \subset\subset \Omega'$ are bounded domains in $\mathbb{R}^3$.

The objective of this section is to find a solution to the above problem. In order to solve (2.1), we solve a sequence of approximate, smooth problems and use a compactness argument to obtain a convergent subsequence. We first define necessary notation and then present the statements of two theorems that will be necessary for our discussion in this section. Then we prove the existence of a solution to (2.1). Finally, we show that if a solution exists, then under certain conditions we can construct a net of problems whose solutions converge to the given solution.

2.1. **Overview of Spaces and Results for the Critical Exponent Problem.** For the remainder of the paper, for a fixed domain $\Omega \subset \mathbb{R}^n$, we denote the standard Sobolev norms on Ω by

$$\|u\|_{L^p} = \left(\int_\Omega |u(x)|^p \, dx \right)^{\frac{1}{p}}, \qquad (2.2)$$
$$\|u\|_{W^{k,p}} = \left(\sum_{i=1}^{k} \|D^i u\|_{L^p}^p \right)^{\frac{1}{p}}.$$

Furthermore, let

$$\text{ess sup } u = \hat{u}, \qquad (2.3)$$
$$\text{ess inf } u = \check{u}.$$

In our subsequent work we will also require regularity conditions on the domain Ω and its boundary. Therefore, we will need the following definition taken from [5]:

Definition 2.1. *A bounded domain $\Omega \subset \mathbb{R}^n$ and its boundary are of class $C^{k,\alpha}$, $0 \leq \alpha \leq 1$, if for each $x_0 \in \partial\Omega$ there is a ball $B(x_0)$ and a one-to-one mapping Ψ of B onto $D \subset \mathbb{R}^n$ such that:*

(1) $\Psi(B \cap \Omega) \subset \mathbb{R}^n_+$,
(2) $\Psi(B \cap \partial\Omega) \subset \partial\mathbb{R}^n_+$,
(3) $\Psi \in C^{k,\alpha}(B),\ \Psi^{-1} \in C^{k,\alpha}(D)$.

We say that a domain Ω is of class C^∞ if for a fixed $0 \leq \alpha \leq 1$ it is of class $C^{k,\alpha}$ for each $k \in \mathbb{N}$. Additionally, for this section and the next we will require the following Theorem and Proposition:

Theorem 2.2. *Suppose $\Omega \subset \mathbb{R}^n$ is a C^∞ domain and assume $f : \overline{\Omega} \times \mathbb{R}^+ \to \mathbb{R}$ is in $C^\infty(\overline{\Omega} \times \mathbb{R}^+)$ and $\rho \in C^\infty(\overline{\Omega})$. Let L be an elliptic operator of the form*

$$Lu = -D_i(a^{ij} D_j u) + cu, \quad \textit{and} \quad a^{ij}, c \in C^\infty(\overline{\Omega}). \qquad (2.4)$$

Suppose that there exist sub- and super-solutions $u_- : \overline{\Omega} \to \mathbb{R}$ *and* $u_+ : \overline{\Omega} \to \mathbb{R}$ *such that the following hold:*

(1) $u_-, u_+ \in C^\infty(\overline{\Omega})$,
(2) $0 < u_-(x) < u_+(x) \quad \forall x \in \overline{\Omega}$.

Then there exists a solution $u \in C^\infty(\overline{\Omega})$ *to*

$$Lu = f(x,u) \quad \text{on } \Omega, \tag{2.5}$$

$$u|_{\partial\Omega} = \rho, \tag{2.6}$$

such that $u_-(x) \le u(x) \le u_+(x)$.

Proposition 2.3. *Let* u *be a solution to a semilinear equation of the form*

$$-\sum_{i,j}^{N} D_i(a^{ij}D_j u) + \sum_{i=1}^{K} b^i u^{n_i} = 0 \text{ in } \Omega, \tag{2.7}$$

$$u|_{\partial\Omega} = \rho, \quad \rho(x) > 0 \ \text{ on } \ \partial\Omega$$

where a^{ij}, b^i *and* $\rho \in C^\infty(\overline{\Omega})$. *Suppose that the semilinear operator in* (2.7) *has the property that* $n_i > 0$ *for some* $1 \le i \le K$. *Let* n_K *be the largest positive exponent and suppose that* $b_K(x) > 0$ *in* $\overline{\Omega}$. *Additionally, assume that one of the following two cases holds:*

$$\text{(1) } n_i < 0 \text{ for some } 1 \le i < K \text{ and if } n_1 = \min\{n_i : n_i < 0\}, \tag{2.8}$$
$$\text{then } b_1(x) < 0 \text{ in } \overline{\Omega}.$$

$$\text{(2) } n_K \text{ is odd and } 0 < n_i \ \text{ for all } 1 \le i \le K. \tag{2.9}$$

If case (2.8) *holds, define*

$$\alpha' = \sup_{c\in\mathbb{R}_+} \left\{ \sum_{i=1}^{K} \sup_{x\in\Omega} b_i(x) b^{n_i} < 0 \quad \forall b \in (0,c) \right\}, \tag{2.10}$$

and let $\alpha = \min\{\alpha', \inf_{x\in\partial\Omega} \rho(x)\}$. *If case* (2.8) *or case* (2.9) *holds, define*

$$\beta' = \inf_{c\in\mathbb{R}} \left\{ \sum_{i=1}^{K} \inf_{x\in\Omega} b_i(x) b^{n_i} > 0 \quad \forall b \in (c,\infty) \right\}, \tag{2.11}$$

and let $\beta = \max\{\beta', \sup_{x\in\partial\Omega} \rho(x)\}$.

Under these assumptions and definitions, if case (2.8) *holds, then* $0 < \alpha \le u \le \beta$. *Otherwise, if case* (2.9) *holds, then* $-\beta' \le u \le \beta$.

For a more detailed statement of Theorem 2.2 and its proof, see Section 4.1. The proof of Proposition 2.3 can be found in Section 5. Now that we have all of the tools we need, we shall now prove the existence of a solution to a problem of the form (2.1).

2.2. **Existence of a Solution to an Ill-Posed Critical Exponent Problem.** For the following discussion, let $\Omega \subset \mathbb{R}^3$ be a closed and bounded domain of C^∞-class. We assume that $\Omega \subset\subset \Omega' \subset \mathbb{R}^3$, with Ω' open and bounded. Here we prove the existence of a solution to the problem

$$-\Delta u + au^5 + bu^i = 0 \quad \text{in } \Omega, \tag{2.12}$$

$$u|_{\partial\Omega} = \rho,$$

where $1 \leq i \leq 4$ is in $\mathbb{N}$,

$$a \in L^p(\Omega'), \quad b \in L^q(\Omega') \cap L^\infty(\Omega'), \quad \frac{6}{4} < p \leq q < \infty, \quad \rho \in W^{1,2}(\Omega'), \tag{2.13}$$

and

$$\check{a} > 0, \quad \hat{b} < 0, \quad \text{and} \quad \check{\rho} > 0. \tag{2.14}$$

Proposition 2.4. *The semilinear problem* (2.12) *has a solution* $u \in W^{1,2}(\Omega)$ *if* $a, b,$ *and* ρ *satisfy the conditions in* (2.13) *and* (2.14).

Proof. To determine a solution to (2.12), we consider the sequence of solutions to the approximate problems

$$\begin{aligned} -\Delta u_n + a_n(u_n)^5 + b_n(u_n)^i = 0 \quad \text{in } \Omega, \\ u_n|_{\partial\Omega} = \rho_n, \end{aligned} \tag{2.15}$$

where $a_n = a * \phi_n$, $b_n = b * \phi_n$, and $\rho_n = \rho * \phi_n$ and $\phi_n = n^3\phi(nx)$ is a positive mollifier where $\int \phi(x)\, dx = 1$. Given that ϕ is a positive mollifier, it is clear that for each $n \in \mathbb{N}$,

$$\check{a}_n > 0, \; \hat{b}_n < 0 \text{ and } \quad \check{\rho}_n > 0.$$

We first verify that the sequence of problems (2.15) has a solution for each n. To do this, we will utilize Theorem 2.2 and Proposition 2.3. Let β'_n and β_n have the same properties as β' and β in Proposition 2.3 for the sequence of problems (2.15). Then using the notation in Proposition 2.3, we can write explicit expressions for β for (2.12), β'_n and β_n. It is not hard to show that

$$\beta = \max\left\{ \left(-\frac{\check{b}}{\check{a}}\right)^{\frac{1}{5-i}}, \hat{\rho} \right\},$$

and

$$\beta'_n = \left(-\frac{\check{b}_n}{\check{a}_n}\right)^{\frac{1}{5-i}} \quad \text{and} \quad \beta_n = \max\left\{\beta'_n, \hat{\rho}_n\right\}.$$

By Proposition 2.3, β'_n and β_n are *a priori* bounds for the approximate problems. Furthermore, it is not difficult to see that for each $n \in \mathbb{N}$ that $-\beta'_n$ and β_n are sub- and super-solutions for (2.15). See Section 5.2 and Theorem 4.3 for more details. Therefore, given that $\rho_n, a_n, b_n \in C^\infty(\overline{\Omega})$ for each n, we have that $u_n \in C^\infty(\overline{\Omega})$ is a solution to (2.15) for each n by Theorem 2.2.

Now observe that for each $n \in \mathbb{N}$, $\beta_n \leq \beta$, which follows from the fact that

$$-b_n(x) = \int (-b(y))\phi_n(x-y)\, dy \leq \int (-\check{b})\phi_n(x-y) = -\check{b}, \tag{2.16}$$

and $a_n(x) \geq \check{a}$, which is verified by a similar calculation. Therefore, by standard L^p elliptic regularity theory

$$\begin{aligned} \|u_n\|_{W^{2,p}} \leq & C(\| - a_n(u_n)^5 - b_n(u_n)^i\|_{L^p} + \|u_n\|_{L^p}) \\ \leq & C(\beta_n^5\|a_n\|_{L^p} + \beta_n^i\|b_n\|_{L^p} + \beta_n) < M < \infty, \end{aligned} \tag{2.17}$$

where M is independent of n given that $\beta_n \leq \beta$, $a_n \to a$ in L^p, $b_n \to b$ in L^q and $p \leq q$. Because $p > \frac{6}{5}$ and Ω is of C^∞-class, $W^{2,p}(\Omega)$ embeds compactly into $W^{1,2}(\Omega)$. Therefore, there exists a convergent subsequence $u_{n_j} \to u$ in $W^{1,2}(\Omega)$. We claim now that u satisfies the following two properties:

(1) $-\beta \leq u \leq \beta$ almost everywhere,
(2) u solves (2.12).

The inequality $-\beta \le u \le \beta$ $a.e.$ follows from the fact the $u_{n_j} \to u$ in $W^{1,2}(\Omega)$ and

$$-\beta \le -\beta_{n_j} \le -\beta'_{n_j} \le u_{n_j} \le \beta_{n_j} \le \beta.$$

Indeed, if we assume that $u > \beta$ on some set of nonzero measure, then for some n the set $A_n = \{x \in \Omega : u(x) > \beta + \frac{1}{n}\}$ has positive measure. Then for all $j \in \mathbb{N}$, we have that

$$\int |u_{n_j} - u|^2 \, dx \ge \int_{A_n} |u_{n_j} - u|^2 \, dx \ge \frac{1}{n^2}\mu(A_n) > 0.$$

But this clearly contradicts the fact that $u_{n_j} \to u$ in $W^{1,2}(\Omega)$. A similar argument shows that $u \ge -\beta,\ a.e$ in Ω.

Finally, we want to show that u solves (2.12). Let $\epsilon > 0$. Then we have that

$$\begin{aligned}\left|\int \left(\nabla u\cdot\nabla v + au^5 v + bu^i v\right)\, dx\right| = \Big|\int &\left(\nabla u\cdot\nabla v + au^5 v + bu^i v\right)\, dx \\ &- \int \left(\nabla u_{n_j}\cdot\nabla v + a_{n_j}(u_{n_j})^5 v + b_{n_j}(u_{n_j})^i v\right)\, dx\Big|,\end{aligned} \tag{2.18}$$

given that u_{n_j} solves (2.15). Then expanding the second line of the above equation we find that

$$\left|\int \nabla u\cdot\nabla v + au^5 v + bu^i v \, dx\right| \tag{2.19}$$

$$\begin{aligned}&\le \int \left|\nabla u\cdot\nabla v - \nabla u_{n_j}\cdot\nabla v\right|\, dx + \int \left|au^5 v - a_{n_j}(u_{n_j})^5 v\right|\, dx \\ &\quad + \int \left|bu^i v - b_{n_j}(u_{n_j})^i v\right|\, dx\end{aligned} \tag{2.20}$$

$$\begin{aligned}&\le \int \left|\nabla u\cdot\nabla v - \nabla u_{n_j}\cdot\nabla v\right|\, dx + \int \left|au^5 v - a(u_{n_j})^5 v\right|\, dx \\ &\quad + \int \left|a(u_{n_j})^5 v - a_{n_j}(u_{n_j})^5 v\right|\, dx + \int \left|bu^i v - b(u_{n_j})^i v\right|\, dx \\ &\quad + \int \left|b(u_{n_j})^i v - b_{n_j}(u_{n_j})^i v\right|\, dx.\end{aligned} \tag{2.21}$$

Every term in (2.21) tends to 0 given that $u_{n_j} \to u$ in $W^{1,2}(\Omega)$, $a_{n_j} \to a$ in $L^p(\Omega)$, $b_{n_j} \to b$ in $L^q(\Omega)$ and $-\beta \le u \le \beta$. To show that the expression

$$\int \left|au^5 v - a(u_{n_j})^5 v\right|\, dx \to 0,$$

we can use a power series expansion to obtain

$$\begin{aligned}\int \left|au^5 v - a(u_{n_j})^5 v\right|\, dx &= \int \left|av\sum_{i=1}^{5}\binom{5}{i}(u_{n_j})^{5-i}(u-u_{n_j})^i\right|\, dx \\ &\le C(5,\beta^4)\sum_{i=1}^{5}\int \left|(u-u_{n_j})av\right|\, dx \\ &\le C\|a\|_{L^p}\|u-u_{n_j}\|_{L^{\frac{2p}{p-1}}}\|v\|_{L^{\frac{2p}{p-1}}} \\ &\le C\|a\|_{L^p}\|u-u_{n_j}\|_{W^{1,2}}\|v\|_{W^{1,2}},\end{aligned}$$

where the last inequality follows from the fact that $W^{1,2}(\Omega)$ embeds into $L^{\frac{2p}{p-1}}(\Omega)$ if $p > \frac{3}{2}$. Finally, by definition of the trace Tu, we have that,

$$Tu = \lim_{j\to\infty} u_{n_j}|_{\partial\Omega} = \lim_{j\to\infty} \rho_{n_j} = \rho,$$

where the limit is taken in $L^2(\partial\Omega)$. Therefore $u - \rho \in W_0^{1,2}(\Omega)$ and so u solves (2.12). $\square$

2.3. **Convergence of Approximate Solutions to an Existing Solution.** In this section we again assume $\Omega \subset \mathbb{R}^3$ is of $C^\infty(\Omega)$-class and that $\Omega \subset\subset \Omega'$, with $\Omega' \subset \mathbb{R}^3$ open and bounded. We also consider the same semilinear problem as in the previous section:

$$-\Delta u + au^5 + bu^i = 0 \quad \text{in } \Omega, \tag{2.22}$$
$$u|_{\partial\Omega} = \rho,$$

and assume that $a \in L^p(\Omega')$, $b \in L^q(\Omega')$, $\frac{6}{5} < p, q < \infty$, $\rho \in W^{1,2}(\Omega')$ and

$$\check{a} > 0,\ \hat{b} < 0, \text{ and } \quad \check{\rho} > 0.$$

Now we assume that a solution $u \in W^{1,2}(\Omega)$ to (2.22) exists, and we consider the convergence of solutions to the following net of approximate problems to u:

$$-\Delta u_\epsilon + a_\epsilon (u_\epsilon)^5 + b_\epsilon (u_\epsilon)^i = 0 \quad \text{in } \Omega,$$
$$u_\epsilon|_{\partial\Omega} = \rho_\epsilon, \tag{2.23}$$

where $a_\epsilon = a * \phi_\epsilon$, $b_\epsilon = b * \phi_\epsilon$, $\rho_\epsilon = \rho * \phi_\epsilon$, $\phi_\epsilon(x) = \frac{1}{\epsilon^3}\phi(\frac{x}{\epsilon})$ and ϕ is a positive mollifier such that $\int \phi(x)\, dx = 1$. Again, given that ϕ is positive, for each $\epsilon < 1$

$$\check{a}_\epsilon > 0, \quad \check{b}_\epsilon < 0, \quad \text{and} \quad \check{\rho}_\epsilon > 0.$$

Because u is a solution to (2.22), Proposition 2.3 implies that

$$-\beta \le -\left(-\frac{\check{b}}{\check{a}}\right)^{\frac{1}{5-i}} \le u \le \max\left\{\hat{\rho}, \left(-\frac{\check{b}}{\check{a}}\right)^{\frac{1}{5-i}}\right\} = \beta.$$

Similarly, for each $\epsilon \in (0,1)$ we have that

$$-\beta_\epsilon \le u_\epsilon \le \max\left\{\hat{\rho}_\epsilon, \left(-\frac{\check{b}_\epsilon}{\check{a}_\epsilon}\right)^{\frac{1}{5-i}}\right\} = \beta_\epsilon.$$

Given that ϕ_ϵ is a positive mollifier, $\beta_\epsilon \le \beta$ for all $\epsilon \in (0,1)$. This follows from (2.16). Therefore $-\beta \le u_\epsilon \le \beta$ for all $\epsilon \in (0,1)$.

Proposition 2.5. *If a solution u to* (2.22) *exists in $W^{1,2}(\Omega)$, then the approximate solutions u_ϵ of* (2.23) *converge to u in $W^{1,2}(\Omega)$ provided that*

$$\max\left\{\hat{\rho}, \left(-\frac{\check{b}}{\check{a}}\right)^{\frac{1}{5-i}}\right\} << 1. \tag{2.24}$$

Proof. Define $\overline{u}_\epsilon$ to be the solution to the following auxiliary problem:

$$-\Delta \overline{u}_\epsilon = -a_\epsilon u^5 - b_\epsilon u^i \quad \text{in } \Omega, \tag{2.25}$$
$$\overline{u}_\epsilon|_{\partial\Omega} = \rho_\epsilon.$$

Note that $\overline{u}_\epsilon$ exists by standard linear, elliptic solution theory because of the bounds on u and the assumption that $p, q \ge \frac{6}{5}$ implies that

$$-a_\epsilon u^5 - b_\epsilon u^i \in H^{-1}(\Omega).$$

Now consider the following string of inequalities:

$$\begin{aligned}\|u-u_\epsilon\|_{W^{1,2}} &\le \|u-\overline{u}_\epsilon\|_{W^{1,2}}+\|\overline{u}_\epsilon-u_\epsilon\|_{W^{1,2}} \\ &\le C\left(\|(a_\epsilon-a)u^5+(b_\epsilon-b)u^i\|_{H^{-1}}\right. \\ &\qquad \left.+C\|a_\epsilon(u_\epsilon)^5+b_\epsilon(u_\epsilon)^i-a_\epsilon u^5-b_\epsilon u^4\|_{H^{-1}}\right).\end{aligned} \tag{2.26}$$

We observe that

$$\|(a_\epsilon-a)u^5+(b_\epsilon-b)u^i\|_{H^{-1}}\le \beta^5\|a_\epsilon-a\|_{L^p}+\beta^i\|b_\epsilon-b\|_{L^q}\to 0, \tag{2.27}$$

for any $1\le p,q<\infty$.

Furthermore, we may rewrite the second term in the inequality using a power series expansion to obtain the following string of inequalities:

$$\begin{aligned}&\|a_\epsilon(u_\epsilon)^5+b_\epsilon(u_\epsilon)^i-a_\epsilon u^5-b_\epsilon u^i\|_{H^{-1}} \\ &\le \|a_\epsilon\sum_{j=1}^{5}\binom{5}{j}(u_\epsilon)^{5-j}(u-u_\epsilon)^j\|_{H^{-1}}+\|b_\epsilon\sum_{j=1}^{i}\binom{i}{j}(u_\epsilon)^{i-j}(u-u_\epsilon)^j\|_{H^{-1}}.\end{aligned} \tag{2.28}$$

Given that $|u-u_\epsilon|\le 2\beta\ a.e.$, for $1<p'<p$, we have

$$\begin{aligned}\|a_\epsilon\sum_{j=1}^{5}\binom{5}{j}(u_\epsilon)^{5-j}(u-u_\epsilon)^j\|_{H^{-1}} &\le \|a_\epsilon\sum_{j=1}^{5}\binom{5}{j}(u_\epsilon)^{5-j}(u-u_\epsilon)^j\|_{L^{p'}} \\ &\le 2^4\beta^4C(5,\Omega)\|a_\epsilon(u-u_\epsilon)\|_{L^{p'}} \\ &\le C(\Omega,5)\beta^4\|a_\epsilon\|_{L^p}\|u-u_\epsilon\|_{L^{\frac{pp'}{p-p'}}} \\ &\le C(\Omega,5)\beta^4\|a_\epsilon\|_{L^p(\Omega)}\|u-u_\epsilon\|_{W^{1,2}},\end{aligned} \tag{2.29}$$

provided that $p>\frac{6}{5}$. Similarly, if $q>\frac{5}{6}$, we have that

$$\|b_\epsilon\sum_{j=1}^{i}\binom{i}{j}(u_\epsilon)^{i-j}(u-u_\epsilon)^j\|_{H^{-1}}\le C(4,\Omega)\beta^{i-1}\|b_\epsilon\|_{L^q}\|u-u_\epsilon\|_{W^{1,2}}. \tag{2.30}$$

Therefore, equations (2.26) - (2.30) imply that

$$\begin{aligned}\|u-u_\epsilon\|_{W^{1,2}}\le \beta^4\|a_\epsilon-a\|_{L^p}+\beta^4\|b_\epsilon-b\|_{L^q}+C(\Omega,5)\beta^4\|a_\epsilon\|_{L^p}\|u-u_\epsilon\|_{W^{1,2}} \\ +C(4,\Omega)\beta^{i-1}\|b_\epsilon\|_{L^q}\|u-u_\epsilon\|_{W^{1,2}}.\end{aligned} \tag{2.31}$$

Given that $\|a_\epsilon-a\|_{L^p}\to 0$ and $\|b_\epsilon-b\|_{L^q}\to 0$, if

$$\beta^4\|a\|_{L^p}<<1, \quad \text{and} \quad \beta^{i-1}\|b\|_{L^q}<<1,$$

we will have that $\|u-u_\epsilon\|_{W^{1,2}}\to 0$. But (2.24) implies the above condition provided that

$$\max\left\{\hat{\rho},\left(-\frac{\check{b}}{\check{a}}\right)^{\frac{1}{5-i}}\right\}<<1,$$

is sufficiently small. □

Now that we have demonstrated the potential that sequences of solutions to approximate problems have in solving classical semilinear equations, we begin to develop our Colombeau Algebra framework. Within this framework, one embeds a problem with rough data into the algebra and obtains a net of solutions. If the net satisfies certain decay estimates, it is regarded as a member of the algebra. Then ideally one can relate the net of solutions to a classical solution using techniques like those discussed in section 2.2 and 2.3.

3. PRELIMINARY MATERIAL: HOLDER SPACES AND COLOMBEAU ALGEBRAS

We now begin to develop the Colombeau Algebra framework that will be used to solve (1.1). We first define Holder Spaces and state precise versions of the classical Schauder estimates given in [12]. The definition of the Colombeau Algebra in which we will be working and these classical elliptic regularity estimates make these spaces the most natural choice in which to do our analysis. Therefore we will work almost exclusively with Holder spaces for the remainder of the paper. Following our discussion of function spaces, we define the Colombeau algebra in which we will work and then formulate an elliptic, semilinear problem in this space.

3.1. **Function Spaces and Norms.** In this paper we will make frequent use of Schauder estimates on Holder spaces defined on an open set $\Omega \subset \mathbb{R}^n$. Here we give notation for the Holder norms and then state the regularity estimates that will be used.

All notation and results are taken from [5]. Assume that $\Omega \subset \mathbb{R}^n$ is open, connected and bounded. Then define the following norms and seminorms:

$$[u]_{\alpha;\Omega} = \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^{\alpha}}, \tag{3.1}$$

$$[u]_{k,0;\Omega} = \sup_{|\beta|=k} \sup_{x\in\Omega} |D^{\beta}u|, \tag{3.2}$$

$$[u]_{k,\alpha;\Omega} = \sup_{|\beta|=k} [D^{\beta}u]_{\alpha;\Omega}, \tag{3.3}$$

$$\|u\|_{C^k(\overline{\Omega})} = |u|_{k;\Omega} = \sum_{j=0}^{k} [u]_{j,0;\Omega}, \tag{3.4}$$

$$\|u\|_{C^{k,\alpha}(\overline{\Omega})} = |u|_{k,\alpha;\Omega} = |u|_{k;\Omega} + [u]_{k,\alpha;\Omega}. \tag{3.5}$$

We interpret $C^{k,\alpha}(\overline{\Omega})$ as the subspace of functions $f \in C^k(\overline{\Omega})$ such that $f^{(k)}$ is α-Holder continuous. Also, we view the subspace $C^{k,\alpha}(\Omega)$ as the subspace of functions $f \in C^k(\Omega)$ such that $f^{(k)}$ is locally $\alpha-$Holder continuous (over compact sets $K \subset\subset \Omega$).

Now we consider the equation

$$Lu = a^{ij}D_{ij}u + b^i uD_iu + cu = f \quad \text{in } \Omega, \tag{3.6}$$

$$u = \rho \quad \text{on } \partial\Omega, \tag{3.7}$$

where L is a strictly elliptic operator satisfying

$$a^{ij} = a^{ji} \quad \text{and} \quad a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2, \quad x \in \Omega, \quad \xi \in \mathbb{R}^n.$$

The following regularity theorems can be found in [5] and [12]. See [5] for proofs. Note that the constant C in the following theorems has no dependence on Λ or λ.

Theorem 3.1. *Assume that Ω is a $C^{2,\alpha}$-class domain in $\mathbb{R}^n$ and that $u \in C^{2,\alpha}(\overline{\Omega})$ is a solution* (3.6)*, where $f \in C^{\alpha}(\overline{\Omega})$ and $\rho \in C^{2,\alpha}(\overline{\Omega})$. Additionally assume that*

$$|a^{ij}|_{0,\alpha;\Omega}, |b^i|_{0,\alpha;\Omega}, |c|_{0,\alpha;\Omega} \leq \Lambda.$$

Then there exists $C > 0$ such that

$$|u|_{2,\alpha;\Omega} \leq C\left(\frac{\Lambda}{\lambda}\right)^3 (|u|_{0;\Omega} + |\rho|_{2,\alpha;\Omega} + |f|_{0,\alpha;\Omega}).$$

This theorem can then be extended to higher order derivatives if one inserts the derivative of a solution u into (3.6) rearranges the equation and repeatedly applies Theorem 3.1. See [12] for details. We summarize this result in the next theorem.

Theorem 3.2. *Let Ω be a $C^{k+2,\alpha}$-class domain and $u \in C^2\Omega \cap C^0(\overline{\Omega})$ be a solution of* (3.6)*, where $f \in C^{k,\alpha}(\overline{\Omega})$ and $\rho \in C^{k+2,\alpha}(\overline{\Omega})$. Additionally assume that*

$$|a^{ij}|_{k,\alpha;\Omega}, |b^i|_{k,\alpha;\Omega}, |c|_{k,\alpha;\Omega} \leq \Lambda.$$

Then $u \in C^{k+2,\alpha;\Omega}(\overline{\Omega})$ and

$$|u|_{k+2,\alpha;\Omega} \leq C^{k+1}\left(\frac{\Lambda}{\lambda}\right)^{3(k+1)}(|u|_{0;\Omega} + |\rho|_{k+2,\alpha;\Omega} + |f|_{k,\alpha;\Omega}),$$

where C is the constant from Theorem 3.1.

3.2. **Colombeau Algebras.** Now that we have defined the basic function spaces that we will be working with and stated the regularity theorems that will be required to obtain necessary decay estimates, we are ready to define the Colombeau algebra with which we will be working and formulate our problem in this algebra.

Let V be a topological vector space whose topology is given by an increasing family of seminorms μ_k. That is, for $u \in V$, $\mu_i(u) \leq \mu_j(u)$ if $i \leq j$. Then letting $I = (0, 1]$, we define the following:

$$\mathcal{E}_V = (V)^I \quad \text{where } u \in \mathcal{E}_V \text{ is a net } (u_\epsilon) \text{ of elements in } V \text{ with } \epsilon \in (0,1], \tag{3.8}$$

$$\mathcal{E}_{M,V} = \{(u_\epsilon) \in \mathcal{E}_V \ | \ \forall k \in \mathbb{N} \ \exists a \in \mathbb{R} \ : \ \mu_k(u_\epsilon) = \mathcal{O}(\epsilon^a) \ \text{ as } \epsilon \to 0\}, \tag{3.9}$$

$$\mathcal{N}_V = \{(u_\epsilon) \in \mathcal{E}_{V,M} \ | \ \forall k \in \mathbb{N} \ \forall a \in \mathbb{R} \ : \ \mu_k(u_\epsilon) = \mathcal{O}(\epsilon^a) \ \text{ as } \epsilon \to 0\}. \tag{3.10}$$

Then the polynomial generalized extension of V is formed by considering the quotient $\mathcal{G}_V = \mathcal{E}_{M,V}/\mathcal{N}_V$. Let's give a few examples, following [12, 6].

Definition 3.3. *If $V = \mathbb{C}$, $r \in \mathbb{C}$, $\mu_k(r) = |r|$, then one obtains $\overline{\mathbb{C}}$, the ring of generalized constants.*

Definition 3.4. *Let $\Omega \subset \mathbb{R}^n$ be an open set, $U_k \subset\subset \Omega$ an exhaustive sequence of compact sets and $\alpha \in \mathbb{N}_0^n$ a multi-index. Then if*

$$V = C^\infty(\Omega), \ \ f \in C^\infty(\Omega), \quad \mu_k(f) = \sup\{|D^\alpha f| \ : \ x \in U_k, \ \ |\alpha| \leq k\},$$

one obtains $\mathcal{G}^s(\Omega)$, the simplified Colombeau Algebra.

Definition 3.5. *If $V = C^\infty(\overline{\Omega})$, where $\Omega \subset \mathbb{R}^n$ is bounded and*

$$\mu_k(f) = \sup\{|D^\alpha f| : |\alpha| \leq k, \ \ x \in \overline{\Omega}\},$$

we denote the generalized extension by $\mathcal{G}(\overline{\Omega})$. The set $\mathcal{E}_{M,C^\infty(\overline{\Omega})}$ will be denoted by $\mathcal{E}_M(\overline{\Omega})$ and be referred to as the space of moderate elements. The set $\mathcal{N}_{C^\infty(\overline{\Omega})}$ will be denoted by $\mathcal{N}(\overline{\Omega})$ and will be referred to as the space of null elements.

Both $\mathcal{G}^s(\Omega)$ and $\overline{\mathbb{C}}$ were developed by Colombeau and laid the basis for the more general construction described in (3.8)-(3.10). See [3] for more details. As in [12], for the purposes of this paper we are concerned with $\mathcal{G}(\overline{\Omega})$ given that we are interested in solving the Dirichlet problem and require a well-defined boundary value. If $(u_\epsilon) \in \mathcal{E}_M(\overline{\Omega})$ is a representative of an element $u \in \mathcal{G}(\overline{\Omega})$, we shall write $u = [(u_\epsilon)]$ to indicate that u is the equivalence class of (u_ϵ). At times we will drop the parentheses and simply write $[u_\epsilon]$. Addition and multiplication of elements in $\mathcal{G}(\overline{\Omega})$ is defined in terms of addition and multiplication of representatives. That is, if $u = [(u_\epsilon)]$ and $v = [(v_\epsilon)]$, then $uv = [(u_\epsilon v_\epsilon)]$

and $u + v = [(u_\epsilon + v_\epsilon)]$. Derivations are defined for $u = [(u_\epsilon)] \in \mathcal{G}(\overline{\Omega})$ by $\partial_{x_i} u = [(\partial_{x_i} u_\epsilon)]$.

Theorem 3.6. *With the above definitions of addition, multiplication and differentiation, $\mathcal{G}(\overline{\Omega})$ is a associative, commutative, differential algebra.*

Proof. This follows from the fact component-wise addition, multiplication, and differentiation makes $V^I = (C^\infty(\overline{\Omega}))^I$ into a differential algebra. By design, $\mathcal{E}_M(\overline{\Omega})$ is a the largest sub-algebra of $(C^\infty(\overline{\Omega}))^I$ that contains $\mathcal{N}(\overline{\Omega})$ as an ideal. Therefore $\mathcal{G}(\overline{\Omega})$ is a differential algebra as well. See [6]. □

Now that we have given the basic definition of a Colombeau algebra, we can discuss how distributions can be embedded into a space of this type.

3.3. **Embedding Schwartz Distributions into Colombeau Algebras.** While the algebras defined above are somewhat unwieldy, these spaces are well suited for analyzing problems with distributional data. The primary reason for this is that the Schwartz distributions $\mathcal{D}'(\Omega)$ can be linearly embedded into them. This allows one to define an *extrinsic* notion of distributional multiplication that is consistent with the pointwise product of C^∞ functions. Here we discuss a method of embedding $\mathcal{D}'(\Omega)$ into $\mathcal{G}^s(\Omega)$ in the event that Ω is a bounded open subset of $\mathbb{R}^n$. Recall that $\mathcal{G}^s(\Omega)$ was defined in (3.4).

The Schwartz distributions on an open set $\Omega \subset \mathbb{R}^n$, denoted $\mathcal{D}'(\Omega)$, are defined to be the dual of $\mathcal{D}(\Omega)$, the space of C^∞ functions with support contained in Ω. For a given $\varphi \in \mathcal{D}(\Omega)$ and $T \in \mathcal{D}'(\Omega)$, the action of T on φ will be denoted by $\langle T, \varphi\rangle$. We now define an embedding of $\mathcal{D}'(\Omega)$ into $\mathcal{G}^s(\Omega)$ and state some of its properties without proof.

We begin by letting $\psi \in \mathcal{S}(\mathbb{R}^n)$, the Schwartz functions, be a function such that $\psi \equiv 1$ on some neighborhood of 0. Then define $\phi \in \mathcal{S}(\mathbb{R}^n)$ by $\phi = \mathcal{F}^{-1}[\psi]$, the inverse Fourier transform of ψ. It is easy to see that

$$\int_{\mathbb{R}^n} \phi \, dx = 1 \quad \text{and} \quad \int_{\mathbb{R}^n} x^\alpha \phi \, dx = 0 \quad \forall |\alpha| \geq 1. \tag{3.11}$$

Let $\phi_\epsilon = \frac{1}{\epsilon^n}\phi(\frac{x}{\epsilon})$ be the usual mollifier.

The properties of ϕ specified in (3.11) are extremely important. In particular, if $\Omega \subset \mathbb{R}^n$ is open and bounded and $\xi \equiv 1$ in a neighborhood of $\overline{\Omega}$, this choice of ϕ as a mollifier makes following map well-defined:

$$\begin{aligned} &i : \mathcal{D}'(\Omega) \to \mathcal{G}^s(\Omega), \\ &i(u) = (((\xi u) * \phi_\epsilon)|_\Omega). \end{aligned} \tag{3.12}$$

We summarize some properties of this map in the following theorem:

Theorem 3.7. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $\xi \equiv 1$ in a neighborhood of $\overline{\Omega}$. Define the map*

$$\begin{aligned} \sigma :&C^\infty(\Omega) \to \mathcal{G}^s(\Omega), \\ &\sigma(f) = (f), \end{aligned} \tag{3.13}$$

where (f) represents a net (u_ϵ) such that $u_\epsilon = f$ for all $\epsilon \in (0,1]$. Then the map (3.12) *is a linear embedding satisfying*

$$i|_{C^\infty(\Omega)} = \sigma. \tag{3.14}$$

Proof. This is a special case of the embedding constructed in [6] in the event that Ω is bounded. □

We will require Theorem 3.7 when we discuss how to solve (1.1) in $\mathcal{G}(\overline{\Omega})$. The approach will be to reformulate the problem (1.1) as a differential equation in $\mathcal{G}(\overline{\Omega})$ by utilizing a slight variation of the above embedding. But before we discuss how to do this, we must first define what we mean by a differential equation in $\mathcal{G}(\overline{\Omega})$.

3.4. **Nets of Semilinear Differential Operators.** We begin by defining a semilinear differential operator on $\mathcal{G}(\overline{\Omega})$. Our construction strongly resembles the construction by Mitrovic and Pilipovic in [12]. For $\epsilon < 1$, if (a_ϵ^{ij}), $(b_\epsilon^i) \in \mathcal{E}_M(\overline{\Omega})$, we obtain a net of operators by defining A_ϵ to be

$$A_\epsilon u_\epsilon = -D_i(a_\epsilon^{ij} D_j u) + \sum_i^K b_\epsilon^i u^{n_i} = -a_\epsilon^{ij} D_i D_j u_\epsilon - (D_i a_\epsilon^{ij})(D_j u_\epsilon) + \sum_{i=1}^K b_\epsilon^i (u_\epsilon)^{n_i},$$

where $n_i \in \mathbb{Z}$. Under certain conditions, we can view a net of operators of the above form as an operator on $\mathcal{G}(\overline{\Omega})$. Here we determine these conditions, which will guarantee that this net of operators is a well-defined operator on $\mathcal{G}(\overline{\Omega})$.

Given an element u in $\mathcal{G}(\overline{\Omega})$, we first need to ensure that $(A_\epsilon u_\epsilon) \in \mathcal{E}_M(\overline{\Omega})$. Based on how derivations and multiplication are defined in $\mathcal{G}(\overline{\Omega})$, the only serious obstacle to this is if $n_i < 0$ for some $i \leq K$. Therefore, we must guarantee that the element $((u_\epsilon)^{n_i})$ is a well-defined representative in $\mathcal{G}(\overline{\Omega})$ if $n_i < 0$. It suffices to ensure that $u = [(u_\epsilon)]$ has an inverse in $\mathcal{G}(\overline{\Omega})$. This is true if for each representative (u_ϵ) of u, there exists $\epsilon_0 \in (0, 1]$ and $m \in \mathbb{N}$ such that for all $\epsilon \in (0, \epsilon_0)$, $\inf_{x \in \overline{\Omega}} |u_\epsilon(x)| \geq C\epsilon^m$. See [6] for more details. So $u \in \mathcal{G}(\overline{\Omega})$ must possess this property in order for the above operator to have any chance of being well-defined. For the rest of this section we assume that u satisfies this condition.

Now suppose $(\overline{a}_\epsilon^{ij})$, $(\overline{b}_\epsilon^i)$ in $\mathcal{E}_M(\overline{\Omega})$, and let

$$\overline{A}_\epsilon u = -\sum_{i,j=1}^N D_i(\overline{a}_\epsilon^{ij} D_j u) + \sum_i^K \overline{b}_\epsilon^i u^{n_i} = \overline{a}_\epsilon^{ij} D_i D_j u_\epsilon - (D_i \overline{a}_\epsilon^{ij})(D_j u_\epsilon) + \sum_{i=1}^K \overline{b}_\epsilon^i (u_\epsilon)^{n_i}.$$

We say that $(A_\epsilon) \sim (\overline{A}_\epsilon)$ if $(a_\epsilon^{ij} - \overline{a}_\epsilon^{ij}), (b_\epsilon^i - \overline{b}_\epsilon^i) \in \mathcal{N}^s(\overline{\Omega})$. Then $(A_\epsilon) \sim (\overline{A}_\epsilon)$ if and only if $(A_\epsilon u_\epsilon - \overline{A}_\epsilon u_\epsilon) \in \mathcal{N}(\overline{\Omega})$ for all $(u_\epsilon) \in \mathcal{E}_M(\overline{\Omega})$ due to the fact that the above operators are linear in (a_ϵ^{ij}) and (b_ϵ^i).

Let $\mathcal{A}$ be the family of nets of differential operators of the above form and define $\mathcal{A}_0 = \mathcal{A}/\sim$. Then for $A \in \mathcal{A}_0$ and $u \in \mathcal{E}_M(\overline{\Omega})$, define

$$A : \mathcal{G}(\overline{\Omega}) \to \mathcal{G}(\overline{\Omega}) \text{ by } Au = [A_\epsilon u_\epsilon],$$

where

$$[A_\epsilon u_\epsilon] = [-a_\epsilon^{ij}][D_i D_j u_\epsilon] + [-D_i a_\epsilon^{ij}][D_j u_\epsilon] + \sum_{i=1}^K [b_\epsilon^i][u_\epsilon^{n_i}]. \tag{3.15}$$

Using this definition, $A \in \mathcal{A}_0$ is a well-defined operator on $\mathcal{G}(\overline{\Omega})$. We summarize this statement in the following proposition.

Proposition 3.8. *$\mathcal{A}_0$ is a well-defined class of differential operators from $\mathcal{G}(\overline{\Omega})$ to $\mathcal{G}(\overline{\Omega})$.*

Proof. Based on the construction of $\mathcal{A}_0$, it is clear that for a given representative (u_ϵ) of $u \in \mathcal{G}(\overline{\Omega})$, $(A_\epsilon u_\epsilon)$ and $(\overline{A}_\epsilon u_\epsilon)$ represent the same element in $\mathcal{G}(\overline{\Omega})$. Furthermore, given a representative (A_ϵ) of $\mathcal{A}_0$, we also have that $[A_\epsilon u_\epsilon] = [A_\epsilon \overline{u}_\epsilon]$ for any two representatives of $u \in \mathcal{G}(\overline{\Omega})$. To see this, we first observe that for each ϵ, every term in $A_\epsilon u_\epsilon$ is linear except for the $(u_\epsilon)^{n_i}$ terms. So to verify the previous statement it suffices to show that for

each $n_i \in \mathbb{Z}$, $((u_\epsilon)^{n_i}) = ((\overline{u}_\epsilon)^{n_i}) + (\overline{\eta}_\epsilon)$, where $(\overline{\eta}_\epsilon) \in \mathcal{N}(\overline{\Omega})$. Given that $[(u_\epsilon)] = [(\overline{u}_\epsilon)]$ in $\mathcal{G}(\overline{\Omega})$, we have $(\overline{u}_\epsilon) = (u_\epsilon) + (\eta_\epsilon)$ for $(\eta_\epsilon) \in \mathcal{N}(\overline{\Omega})$. For fixed ϵ, $n_i \in \mathbb{Z}^+$,

$$(\overline{u}_\epsilon)^{n_i} = (u_\epsilon + \eta_\epsilon)^{n_i} = \sum_{j=0}^{n_i} \binom{n_i}{j} (u_\epsilon)^j (\eta_\epsilon)^{n_i - j} = (u_\epsilon)^{n_i} + \overline{\eta}_\epsilon,$$

where $\overline{\eta}_\epsilon$ consists of the summands that each contain some nonzero power of η_ϵ. Clearly the net $(\overline{\eta}_\epsilon) \in \mathcal{N}(\overline{\Omega})$. If $n_i \in \mathbb{Z}^-$, then for a fixed ϵ,

$$(\overline{u}_\epsilon)^{n_i} = \frac{1}{(u_\epsilon + \eta_\epsilon)^{|n_i|}} = \frac{1}{\sum_{j=0}^{|n_i|} \binom{|n_i|}{j} (u_\epsilon)^j (\eta_\epsilon)^{|n_i| - j}} = \frac{1}{(u_\epsilon)^{|n_i|} + \overline{\eta}_\epsilon}.$$

By looking at the difference

$$(u_\epsilon)^{n_i} - \frac{1}{(u_\epsilon)^{|n_i|} + \overline{\eta}_\epsilon} = \frac{\overline{\eta}_\epsilon}{((u_\epsilon)^{|n_i|})((u_\epsilon)^{|n_i|} + \overline{\eta}_\epsilon)} = \hat{\eta}_\epsilon,$$

we see that the net $((u_\epsilon)^{n_i}) = ((\overline{u}_\epsilon)^{n_i}) + (\hat{\eta}_\epsilon)$, where $(\hat{\eta}_\epsilon) \in \mathcal{N}(\overline{\Omega})$. Therefore for any $u \in \mathcal{G}(\overline{\Omega})$ possessing an inverse, and any $A \in \mathcal{A}_0$, the expression $Au = [A_\epsilon u_\epsilon] \in \mathcal{G}(\overline{\Omega})$ is well-defined. $\square$

3.5. **The Dirichlet Problem in $\mathcal{G}(\overline{\Omega})$.** Using the above definition of $\mathcal{A}$, we can now define our semilinear Dirichlet problem on $\mathcal{G}(\overline{\Omega})$. Let $u, \rho \in \mathcal{G}(\overline{\Omega})$ where $\Omega \subset \mathbb{R}^n$ is open, bounded and of C^∞-class. Then let E be a total extension operator of Ω such that for $f \in C^\infty(\Omega)$, $Ef \in C^\infty(\mathbb{R}^n)$ and $Ef|_\Omega = f$. See [1] for details. Using E we may may define $u|_{\partial\Omega} = \rho|_{\partial\Omega}$ for elements $u, \rho \in \mathcal{G}(\overline{\Omega})$ if there are representatives (u_ϵ) and (ρ_ϵ) such that

$$u_\epsilon|_{\partial\Omega} = \rho_\epsilon|_{\partial\Omega} + n_\epsilon|_{\partial\Omega},$$

where n_ϵ is a net of C^∞ functions defined in a neighborhood of $\partial\Omega$ such that

$$\sup_{x \in \partial\Omega} |n_\epsilon(x)| = o(\epsilon^a) \quad \forall a \in \mathbb{R}. \tag{3.16}$$

This will ensure that $u|_{\partial\Omega} = \rho|_{\partial\Omega}$ does not depend on representatives [12]. From now on we will abbreviate $u|_{\partial\Omega} = \rho|_{\partial\Omega}$ by $u|_{\partial\Omega} = \rho$. With this definition of boundary equivalence, for a given operator $A \in \mathcal{A}_0$, the Dirichlet problem

$$\begin{aligned} Au &= 0 \quad \text{in } \Omega, \\ u|_{\partial\Omega} &= \rho, \end{aligned} \tag{3.17}$$

is well-defined in $\mathcal{G}(\overline{\Omega})$. Now we state the conditions under which the above problem can be solved in $\mathcal{G}(\overline{\Omega})$.

4. Overview of the Main Results

We begin this section by stating the main existence result for the Dirichlet problem (3.17). Let $A \in \mathcal{A}_0$ be an operator on $\mathcal{G}(\overline{\Omega})$ defined by (3.15). Also assume that the coefficients of A have representatives $(a^{ij}_\epsilon), (b^i_\epsilon) \in \mathcal{E}_M(\overline{\Omega})$ that satisfy the following properties for $\epsilon \in (0,1)$:

$$\begin{aligned} a^{ij}_\epsilon = a^{ji}_\epsilon, &\quad a^{ij}_\epsilon \xi_i \xi_j \geq \lambda_\epsilon |\xi|^2 \geq C_1 \epsilon^a |\xi|^2, \\ \text{for each } k \in \mathbb{N}, &\quad |a^{ij}_\epsilon|_{k+1,\alpha;\Omega}, \; |b^i_\epsilon|_{k,\alpha;\Omega} \leq \Lambda_{k,\epsilon} \leq C_2 \epsilon^b, \\ b^1_\epsilon \leq -C_3 \epsilon^c, &\quad \{n_i : n_i < 0\} \neq \emptyset, \quad n_1 = \min\{n_i : n_i < 0\} \\ b^K_\epsilon \geq C_4 \epsilon^d, &\quad \{n_i : n_i > 0\} \neq \emptyset \quad n_K = \max\{n_i : n_i > 0\}, \end{aligned} \tag{4.1}$$

where C_1, C_2, C_3 and C_4 are positive constants independent of ϵ. Then the following Dirichlet problem has a solution in $\mathcal{G}(\overline{\Omega})$:

$$Au = [A_\epsilon u_\epsilon] = 0 \text{ in } \Omega, \qquad (4.2)$$
$$u|_{\partial\Omega} = \rho.$$

We summarize this result in the following theorem, which will be the focus of the remainder of the paper:

Theorem 4.1. *Suppose that $A : \mathcal{G}(\overline{\Omega}) \to \mathcal{G}(\overline{\Omega})$ is in $\mathcal{A}_0$ and that the conditions of* (4.1) *hold. Assume that for each $\epsilon \in (0,1]$ that for some $1 \leq i \leq K$, b^i_ϵ is non-constant. Furthermore, assume that $\rho \in \mathcal{G}(\overline{\Omega})$ has a representative (ρ_ϵ) such that for $\epsilon < 1$, $\rho_\epsilon \geq C\epsilon^a$ for some $C > 0$ and $a \in \mathbb{R}$. Then there exists a solution to the Dirichlet problem* (4.2) *in $\mathcal{G}(\overline{\Omega})$.*

Proof. The proof will be given in Section 6. □

Remark 4.2. *We can actually weaken the assumptions in* (4.1) *so that the conditions on the representatives $(a^{ij}_\epsilon), (b^1_\epsilon), (b^K_\epsilon), (\rho_\epsilon)$ only have to hold for all $\epsilon \in (0, \epsilon_0)$ for some $\epsilon_0 \in (0,1)$. Suppose that this is the case, and that using these conditions we are able to show that for all $\epsilon \in (0, \epsilon_0)$, there exists u_ϵ that solves*

$$A_\epsilon u_\epsilon = 0 \quad \text{in } \Omega, \qquad (4.3)$$
$$u_\epsilon|_{\partial\Omega} = \rho_\epsilon.$$

If u_ϵ satisfies the additional property that for all $k \in \mathbb{N}$, there exists some $\epsilon'_0 \in (0, \epsilon_0)$, $C > 0$, and $a \in \mathbb{R}$ such that for all $\epsilon \in (0, \epsilon'_0)$, $|u_\epsilon|_{k,\alpha} \leq C\epsilon^a$, then we can form a solution $(v_\epsilon) \in \mathcal{E}_M(\overline{\Omega})$ to (4.2) *by defining $v_\epsilon = u_\epsilon$ for $\epsilon \in (0, \epsilon_0)$ and $v_\epsilon = u_{\epsilon_0}$ for $\epsilon \in [\epsilon_0, 1]$. The solution theory that we develop to prove Theorem 4.1 with the stronger conditions* (4.1) *will also imply the existence of the partial net (u_ϵ) of solutions to* (4.3) *in the event that the constraints outlined in* (4.1) *only hold for $\epsilon \in (0, \epsilon_0) \subset (0,1)$. We will require this fact when we consider how to embed and solve* (1.1) *in $\mathcal{G}(\overline{\Omega})$ later on in Section 4.3.*

We begin assembling the tools we will need to prove Theorem 4.1. The first tool we need is a method capable of solving a large class of semilinear problems. The method of sub- and super-solutions meets this need, and we discuss this process of solving elliptic, semilinear problems in the following section.

4.1. **The Method of Sub- and Super-Solutions.** In Theorem 4.3 below, we state a fixed point result that will be essential in proving Theorem 4.1. This fixed point result is known as the method of sub- and super-solutions due to the fact that for a given operator A, the method relies on finding a sub-solution u_- and super-solution u_+ such that $u_- < u_+$. A large part of this paper is devoted to finding a net of positive sub- and super-solutions for (4.2) and establishing growth conditions for them. In the proof below, let

$$Lu = -D_i(a^{ij}D_j u) + cu, \qquad (4.4)$$

be an elliptic operator where

$$a^{ij} = a^{ji}, \quad a^{ij}\xi_i\xi_j \geq \lambda|\xi|^2 \quad \text{and} \quad a^{ij}, c \in C^\infty(\overline{\Omega}).$$

We now state and prove the sub- and super-solution fixed point result for these assumptions.

Theorem 4.3. *Suppose $\Omega \subset \mathbb{R}^n$ is a C^∞ domain and assume $f : \overline{\Omega} \times \mathbb{R}^+ \to \mathbb{R}$ is in $C^\infty(\overline{\Omega} \times \mathbb{R}^+)$ and $\rho \in C^\infty(\overline{\Omega})$. Let L be of the form* (4.4)*. Suppose that there exist functions $u_- : \overline{\Omega} \to \mathbb{R}$ and $u_+ : \overline{\Omega} \to \mathbb{R}$ such that the following hold:*

(1) $u_-, u_+ \in C^\infty(\overline{\Omega})$,
(2) $0 < u_-(x) \leq u_+(x) \quad \forall x \in \overline{\Omega}$,
(3) $Lu_- \leq f(x, u_-)$,
(4) $Lu_+ \geq f(x, u_+)$,
(5) $u_- \leq \rho$ *on* $\partial\Omega$,
(6) $u_+ \geq \rho$ *on* $\partial\Omega$.

Then there exists a solution u *to*

$$\begin{aligned} Lu &= f(x,u) \quad \textit{on } \Omega, \\ u|_{\partial\Omega} &= \rho, \end{aligned} \tag{4.5}$$

such that

(i) $u \in C^\infty(\overline{\Omega})$,
(ii) $u_-(x) \leq u(x) \leq u_+(x)$.

Proof. The general approach of the proof will be to construct a monotone sequence $\{u_n\}$ that is point-wise bounded above and below by our super- and sub-solutions, u_+ and u_-. We will then apply elliptic regularity estimates and the Arzela-Ascoli Theorem to conclude that the sequence $\{u_n\}$ has a $C^\infty(\overline{\Omega})$ limit u that is a solution to

$$\begin{aligned} Lu &= f(x,u) \quad \text{on } \Omega, \\ u|_{\partial\Omega} &= \rho. \end{aligned} \tag{4.6}$$

Given that $u_-(x), u_+(x) \in C^\infty(\overline{\Omega})$, the interval $[\min u_-(x), \max_+ u_+(x)] \subset \mathbb{R}^+$ is well-defined. We then restrict the domain of the function f to the compact set $K = \overline{\Omega} \times [\min u_-(x), \max_+ u_+(x)]$. Given that $f \in C^\infty(\overline{\Omega} \times \mathbb{R}^+)$, it is clearly in $C^\infty(\overline{\Omega} \times [\min u_-(x), \max_+ u_+(x)])$ and so the function $|\frac{\partial f(x,t)}{\partial t}|$ is continuous and attains a maximum on K. Denoting this maximum value by m, let $M = \max\{m, -\inf_{x\in\overline{\Omega}} c(x)\}$. Then consider the operator

$$Au = Lu + Mu,$$

and the function

$$F(x,t) = Mu + f(x,t).$$

Note that this choice of M ensures that $F(x,t)$ is an increasing function in t on K and that A is an invertible operator. Also, we clearly have the following:

$$A(u) = F(x,u) \Longleftrightarrow Lu = f(x,u), \tag{4.7}$$

$$A(u_-) \leq F(x,u_-) \Longleftrightarrow L(u_-) \leq f(x,u_-), \tag{4.8}$$

$$A(u_+) \geq F(x,u_+) \Longleftrightarrow L(u_+) \geq f(x,u_+). \tag{4.9}$$

The first step in the proof is to construct the sequence $\{u_n\}$ iteratively. Let u_1 satisfy the equation

$$\begin{aligned} A(u_1) &= F(x,u_-) \quad \text{on } \Omega, \\ u_1|_{\partial\Omega} &= \rho. \end{aligned} \tag{4.10}$$

We observe that for $u, v \in H^1_0(\Omega)$, the operator A satisfies

$$C_1\|u\|^2_{H^1(\Omega)} \leq \langle Au, u\rangle\,, \qquad \text{and} \qquad \langle Au, v\rangle \leq \|u\|^2_{H^1(\Omega)}\|v\|^2_{H^1(\Omega)},$$

where

$$\langle u, v\rangle = \int_\Omega uv dx, \qquad \text{and} \qquad \langle Lu, v\rangle = \int_\Omega (a^{ij}D_j u D_i v + cuv) dx.$$

Therefore the Lax-Milgram theorem implies that there exists a weak solution $u_1 \in H^1(\Omega)$ satisfying $u_1 - \rho \in H^1_0(\Omega)$. Given our assumptions on $F(x,t)$ and ρ, $F(x,u_+) \in H^m(\Omega)$ and $\rho \in H^m(\Omega)$ for all $m \in \mathbb{N}$. Therefore, by standard elliptic regularity arguments, $u_1 \in H^m(\Omega)$ for all $m \in \mathbb{N}$. This, the assumption that Ω is of C^∞-class and the assumption that $a^{ij}, c, \rho \in C^\infty(\overline{\Omega})$ imply that $u_1 \in C^\infty(\overline{\Omega})$ and $u_1 = \rho$ on $\partial\Omega$. Therefore, we may iteratively define the sequence $\{u_j\} \subset C^\infty(\overline{\Omega})$ where

$$\begin{aligned} A(u_j) &= F(x, u_{j-1}) \quad \text{on } \Omega, \\ u_j|_{\partial\Omega} &= \rho. \end{aligned} \tag{4.11}$$

The next step is to verify that the sequence $\{u_j\}$ is a monotonic increasing sequence satisfying $u_- \leq u_1 \leq \cdots \leq u_{j-1} \leq u_j \leq \cdots \leq u_+$. We prove this by induction. First we observe that

$$\begin{aligned} A(u_- - u_1) \leq F(x,u_-) - F(x,u_-) = 0 \quad \text{on } \Omega, \\ (u_- - u_1)|_{\partial\Omega} \leq 0. \end{aligned} \tag{4.12}$$

Therefore, by the weak maximum principle, $u_- \leq u_1$ on $\overline{\Omega}$. Now suppose that $u_{j-1} \leq u_j$. Then

$$\begin{aligned} A(u_j - u_{j+1}) = F(x,u_{j-1}) - F(x,u_j) \leq 0 \quad \text{on } \Omega, \\ (u_j - u_{j+1})|_{\partial\Omega} = 0. \end{aligned} \tag{4.13}$$

given that $F(x,t)$ is an increasing function in the variable t and $u_{j-1} \leq u_j$. The weak maximum principle again implies that $u_j \leq u_{j+1}$, so by induction we have that $\{u_j\}$ is monotonic increasing sequence that is point-wise bounded below by $u_-(x)$. Now we show that our increasing sequence is point-wise bounded above by $u_+(x)$ by proceeding in a similar manner. Given that $u_- \leq u_+$ and u_+ is a super-solution, we have that

$$\begin{aligned} A(u_1 - u_+) \leq F(x,u_-) - F(x,u_+) \leq 0 \quad \text{on } \Omega, \\ (u_1 - u_+)|_{\partial\Omega} \leq 0. \end{aligned} \tag{4.14}$$

The weak maximum principle implies that $u_1 \leq u_+$. Now assume that $u_j \leq u_+$. Then

$$\begin{aligned} A(u_{j+1} - u_+) \leq F(x,u_j) - F(x,u_+) \leq 0 \quad \text{on } \Omega, \\ (u_{j+1} - u_+)|_{\partial\Omega} \leq 0, \end{aligned} \tag{4.15}$$

given that $F(x,t)$ is an increasing function and $u_j \leq u_+$. So by induction the sequence $\{u_j\}$ is a monotonic increasing sequence that is point-wise bounded above by $u_+(x)$ and point-wise bounded below by $u_-(x)$.

Up to this point, we have constructed a monotonic increasing sequence $\{u_j\} \subset C^\infty(\overline{\Omega})$ such that for each j, u_j satisfies the Dirichlet problem (4.11) and is point-wise bounded below by u_- and above by u_+. The next step will be to apply the Arzela-Ascoli theorem and a bootstrapping argument to conclude that this sequence converges to $u \in C^\infty(\overline{\Omega})$. We first show that it converges to $u \in C(\overline{\Omega})$ by an application of the Arzela-Ascoli Theorem. Clearly the family of functions $\{u_j\}$ is point-wise bounded, so it is only necessary to establish the equicontinuity of the sequence. Given that each function u_j solves the problem (4.11), by standard L^p elliptic regularity estimates (cf. [5]) we have that

$$\|u_j\|_{W^{2,p}} \leq C(\|u_j\|_{L^p} + \|F(x,u_{j-1})\|_{L^p}).$$

The regularity of $F(x,t)$ and the sequence $\{u_j\}$ along with the above estimate and the compactness of $\overline{\Omega} \times [\inf u_-, \sup u_+]$ imply that there exists a constant N such that

$\|F(x, u_{j-1})\|_{L^p} \leq N$ for all j. Therefore, if $p > 3$, the above bound and the fact that $u_- \leq u_j \leq u_+$ imply that for each $j \in \mathbb{N}$,

$$|u_j|_{1,\alpha;\Omega} \leq C\|u\|_{W^{2,p}} \leq \infty,$$

where $\alpha = 1 - \frac{3}{p}$. This implies that the sequence $\{u_j\}$ is equicontinuous. The Arzela-Ascoli Theorem then implies that there exists a $u \in C(\overline{\Omega})$ and a subsequence $\{u_{j_k}\}$ such that $u_{j_k} \to u$ uniformly. Furthermore, due to the fact that the sequence $\{u_j\}$ is monotonic increasing, we actually have that $u_j \to u$ uniformly on $\overline{\Omega}$. Once we have that $u_j \to u$ in $C(\overline{\Omega})$, we apply L^p regularity theory again to conclude that

$$\begin{aligned} |u_j - u_k|_{1,\alpha;\Omega} \leq & C\|u_j - u_k\|_{W^{2,p}} \\ \leq & C'(\|u_j - u_k\|_{L^p} + \|F(x, u_{j-1}) - F(x, u_{k-1})\|_{L^p}). \end{aligned} \tag{4.16}$$

Note that the above estimate follows from the fact that $u_{j_k+1} - u_{j_l+1}$ satisfies

$$\begin{aligned} A(u_j - u_k) &= F(x, u_{j-1}) - F(x, u_{k-1}) \quad \text{on } \Omega, \\ (u_j - u_k)|_{\partial\Omega} &= 0. \end{aligned} \tag{4.17}$$

Given that $u_j \to u$ in $C(\overline{\Omega})$, (4.16) implies that the sequence $\{u_j\}$ is a Cauchy sequence in $C^1(\overline{\Omega})$. The completeness of $C^1(\overline{\Omega})$ then implies that this subsequence has a limit $v \in C^1(\overline{\Omega})$, and given that $u_j \to u$ in $C(\overline{\Omega})$, it follows that $u = v$. Similarly, by repeating the above argument and using higher order L^p estimates we have that

$$\begin{aligned} |u_j - u_k|_{2,\alpha;\Omega} \leq & C(\|u_j - u_k\|_{W^{3,p}}) \\ \leq & C'(\|u_j - u_k\|_{W^{1,p}} + \|F(x, u_{j-1}) - F(x, u_{k-1})\|_{W^{1,p}}), \end{aligned} \tag{4.18}$$

where $u_j \to u$ in $C^1(\overline{\Omega})$ as $k \to \infty$. Again, (4.18), the regularity of F and the fact that $u_j \to u$ in $C^1(\overline{\Omega})$ imply that the sequence $\{u_j\}$ is Cauchy in $C^2(\overline{\Omega})$. A simple induction argument then shows that $u \in C^\infty(\overline{\Omega})$.

The final step of the proof is to show that u is an actual solution to the problem (4.5). It suffices to show that u is a weak solution to the above problem. It is clear that $u = \rho$ on $\partial\Omega$, so we only need to show that u satisfies (4.5) on Ω. Fix $v \in H^1_0(\Omega)$. Then based on the definition of the sequence $\{u_j\}$, we have

$$\int_\Omega (a^{ij} D_j u_j D_i v + M u_j v) dx = \int_\Omega (f(x, u_{j-1}) + M u_{j-1}) v dx.$$

As $u_j \to u$ uniformly in $C(\overline{\Omega})$, we let let $j \to \infty$ to conclude that

$$\int_\Omega (a^{ij} D_j u D_i v + M u v) dx = \int_\Omega (f(x, u) + M u) v dx.$$

Upon canceling the term involving M from both sides, we find that u is a weak solution. $\square$

4.2. **Outline of the Proof of Theorem 4.1.** Now that the sub- and super-solution fixed point theorem is in place, we give an outline for how to prove Theorem 4.1.

Step 1: *Formulation of the problem.* We phrase (4.2) in a way that allows us to solve a net of semilinear elliptic problems. We assume that the coefficients of A and boundary data ρ have representatives (a^{ij}_ϵ), (b^i_ϵ), and (ρ_ϵ) in $\mathcal{E}_M(\overline{\Omega})$ satisfying the

assumptions (4.1). Then for this particular choice of representatives, we solve the family of problems:

$$A_\epsilon u_\epsilon = -\sum_{i,j=1}^{N} D_i(a_\epsilon^{ij} D_j u_\epsilon) + \sum_{i}^{N} b_\epsilon^i u_\epsilon^{n_i} = 0 \quad \text{in } \Omega, \tag{4.19}$$
$$u_\epsilon|_{\partial\Omega} = \rho_\epsilon.$$

Then we must ensure that the net of solutions $(u_\epsilon) \in \mathcal{E}_M(\overline{\Omega})$ and ensure that (4.19) is satisfied for other representatives of A, ρ, u.

Step 2: *Determine L^∞-estimates and a net of generalized constant sub-solutions and super-solutions*. We determine constant, *a priori* L^∞ bounds such that for a positive net of solutions (u_ϵ) of the semilinear problem (4.19), there exist constants $a_1, a_2 \in \mathbb{R}$, $C_1, C_2 > 0$ independent of $\epsilon \in (0,1)$ such that

$$C_1 \epsilon^{a_1} < \alpha_\epsilon \leq u_\epsilon \leq \beta_\epsilon < C_2 \epsilon^{a_2}.$$

These estimates are constructed in such a way that for each ϵ, the pair $\alpha_\epsilon, \beta_\epsilon$ are sub- and super-solutions for (4.19).

Step 3: *Apply fixed point theorem to solve each semilinear problem in* (4.19). Using the sub- and super-solutions $\alpha_\epsilon, \beta_\epsilon$, we apply Theorem 4.3 to obtain a net of solutions $(u_\epsilon) \in C^\infty(\overline{\Omega})$.

Step 4: *Verify that the net of solutions $(u_\epsilon) \in \mathcal{E}_M(\overline{\Omega})$*. Here we show that the net of solutions satisfies the necessary growth conditions in ϵ using the growth conditions on the sub- and super- solutions and Theorem 3.1.

Step 5: *Verify that the solution is well-defined*. Once we've determined that the net of solutions $(u_\epsilon) \in \mathcal{E}_M(\overline{\Omega})$, we conclude that $[(u_\epsilon)] \in \mathcal{G}(\overline{\Omega})$ is a solution to the Dirichlet problem (4.2) by showing that the solution is independent of the representatives chosen. Note that most of the work for this step was done in Proposition 3.8.

We shall carry out the above steps in our proof of Theorem 4.1 in Section 6. We still need to determine a net of sub- and super- solutions for (4.1), which we do in Section 5. But before we move on to this and the other steps in the above outline, we briefly return to the motivating problem (1.1) by discussing how to embed a problem with distributional data into $\mathcal{G}(\overline{\Omega})$.

4.3. **Embedding a Semilinear Elliptic PDE with Distributional Data into $\mathcal{G}(\overline{\Omega})$.** Now that we have defined what it means to solve a differential equation in $\mathcal{G}(\overline{\Omega})$, we are ready to return to the problem discussed at the beginning of the paper. We are interested in solving an elliptic, semilinear Dirichlet problem of the form

$$-\sum_{i,j=1}^{N} D_i(a^{ij} D_j u) + \sum_{i=1}^{K} b^i u^{n_i} = 0 \quad \text{in } \Omega, \tag{4.20}$$
$$u|_{\partial\Omega} = \rho,$$

where a^{ij}, b^i and ρ are potentially distributional and $n_i \in \mathbb{Z}$ for each i. If we can formulate this problem as a family of equations similar to (4.19), then it can readily be solved in $\mathcal{G}(\overline{\Omega})$ by Theorem 4.1. What we require is a method of embedding our problem into $\mathcal{G}(\overline{\Omega})$. The key to constructing such an embedding will be Theorem 3.7.

For the following discussion, let $\Omega \subset\subset \Omega' \subset \mathbb{R}^n$ be open and bounded. Additionally assume that Ω is of C^∞-class. Given a distribution $u \in \mathcal{D}'(\Omega')$, define the restriction $u|_\Omega$

to be u restricted to test functions $\rho \in \mathcal{D}(\Omega)$. Let $\mathcal{D}'(\Omega')|_\Omega \subset \mathcal{D}'(\Omega)$ be the space of distributions obtained in this way. Additionally, given an element $v \in \mathcal{G}^s(\Omega')$, define

$$v|_{\overline{\Omega}} = (v_\epsilon|_{\overline{\Omega}}) + \mathcal{N}(\overline{\Omega}).$$

Then we have the following proposition:

Proposition 4.4. *Let i denote the embedding defined in Theorem 3.7. Define the following map*

$$\begin{aligned} \hat{\sigma} : & C^\infty(\overline{\Omega}) \to \mathcal{G}(\overline{\Omega}), \\ & \hat{\sigma}(f) = (f), \end{aligned} \tag{4.21}$$

where (f) is a net (u_ϵ) such that $u_\epsilon = f$ for all $\epsilon \in (0,1]$. Then

$$\begin{aligned} \hat{i} : \mathcal{D}'(\Omega')|_\Omega \to \mathcal{G}(\overline{\Omega}), \\ \hat{i}(u|_\Omega) = (i(u)|_{\overline{\Omega}}), \end{aligned} \tag{4.22}$$

is a linear embedding such that $\hat{i}|_{C^\infty(\overline{\Omega})} = \hat{\sigma}$.

Proof. That $\hat{i}$ is linear and well-defined follows from the properties of restriction maps and the fact that i is linear and injective. To verify that $\hat{i}|_{C^\infty(\overline{\Omega})} = \hat{\sigma}$, we use that fact that Ω is of C^∞-class, which implies that there exists a total extension operator E such that

$$E : C^\infty(\overline{\Omega}) \to C^\infty(\mathbb{R}^n), \tag{4.23}$$

$$(Ef)|_\Omega = f \quad \text{for each } f \in C^\infty(\Omega). \tag{4.24}$$

See [1] for details. Then for each $f \in C^\infty(\overline{\Omega})$, $f = (Ef)|_\Omega$. Clearly $Ef \in \mathcal{D}'(\Omega')$, and therefore, $C^\infty(\overline{\Omega}) \subset \mathcal{D}'(\Omega')|_\Omega$. The fact that $\hat{i}|_{C^\infty(\overline{\Omega})} = \hat{\sigma}$ then follows from Theorem 3.7 and the definition of i. Finally, we need to show that $\hat{i}$ is injective. Suppose that $\hat{i}(u) = 0$ for some $u \in \mathcal{D}'(\Omega')|_\Omega$. Then if ϕ_ϵ and ξ are the same as in Theorem 3.7, we have that

$$(i(u)|_{\overline{\Omega}}) = (((\xi u) * \phi_\epsilon)|_{\overline{\Omega}}) \in \mathcal{N}(\overline{\Omega}).$$

This implies that

$$((\xi u) * \phi_\epsilon) \to 0 \quad \text{uniformly on } \overline{\Omega}.$$

Therefore, for any $\varphi \in \mathcal{D}(\Omega)$, we have that

$$\langle u, \varphi \rangle = \langle u, \xi\varphi \rangle = \langle \xi u, \varphi \rangle = \lim_{\epsilon \to 0} \langle (\xi u) * \phi_\epsilon, \varphi \rangle = 0. \tag{4.25}$$

So $\hat{i}$ is injective. □

Now suppose that we are given a problem of the form (4.20), where the terms a^{ij}, b^i and ρ are in $\mathcal{D}'(\Omega')$. Then we may use Proposition 4.4 to embed the coefficients a^{ij}, b^i and ρ into $\mathcal{G}(\overline{\Omega})$. We will denote a representative of the image of each these terms in $\mathcal{G}(\overline{\Omega})$ by $(a^{ij}_\epsilon), (b^i_\epsilon)$ and (ρ_ϵ). Then for a choice of representatives, we obtain a net of problems of the form (4.19).

In order to solve this net of problems by using Theorem 4.1, we need there to exist a choice of representatives $(a^{ij}_\epsilon), (b^i_\epsilon)$ and (ρ_ϵ) that satisfy the conditions specified in (4.1). While these conditions might seem exacting, this solution framework still admits a wide range of interesting problems. This is evident when one considers the following proposition:

Proposition 4.5. *Let $n_i \in \mathbb{Z}$ be a collection of integers for $1 \leq i \leq K$. Assume that there exist $1 \leq i, j \leq K$ such that $n_i < 0$ and $n_j > 0$ and let*

$$n_1 = \min\{n_i : \ n_i < 0\}, \quad \textit{and} \quad n_K = \max\{n_i : \ n_i > 0\}.$$

Suppose that $a^{ij}, b^1, b^K, \rho \in C(\Omega')$ and $b^2, \cdots, b^{K-1} \in \mathcal{D}'(\Omega')$, where $\Omega' \subset \mathbb{R}^n$ is an open and bounded set. Additionally assume that a^{ij} satisfies the symmetric, ellipticity condition and $\rho > 0$, $b_1 < 0$ and $b_K > 0$ in Ω'. Then if $\Omega \subset\subset \Omega'$ is of C^∞-class, the problem

$$-\sum_{i,j=1}^{N} D_i(a^{ij} D_j u) + \sum_{i=1}^{K} b^i u^{n_i} = 0 \quad \textit{in } \Omega, \tag{4.26}$$
$$u|_{\partial\Omega} = \rho,$$

admits a solution in $\mathcal{G}(\overline{\Omega})$.

Proof. This follows from Proposition 4.4, Theorem 4.1, Remark 4.2 and the fact that $(a^{ij} * \phi_\epsilon)$, $(b^1 * \phi_\epsilon)$, $(b^K * \phi_\epsilon)$ and $(\rho * \phi_\epsilon)$ converge uniformly to a^{ij}, b^1, b^K and ρ in $\overline{\Omega}$. For ϵ sufficiently small, the corresponding problem (4.19) in $\mathcal{G}(\overline{\Omega})$ will satisfy the conditions specified in (4.1). Therefore, Theorem 4.1 and Remark 4.2 imply the result. $\square$

With the issue of solving (4.20) at least partially resolved, we return to the task of proving Theorem 4.1. We begin by establishing some *a priori* L^∞-bounds for a solution to our semilinear problem (4.26) if the given data is smooth.

5. Sub- and Super-Solution Construction and Estimates

Given an operator $A \in \mathcal{A}_0$ with coefficients satisfying (4.1), our solution strategy for the Dirichlet problem (4.2) is to solve the family of problems (4.19) and then establish the necessary decay estimates. In order for this to be a viable strategy, we first need to show that (4.19) has a solution for each $\epsilon \in (0, 1)$. Given that $n_i < 0$ for some $1 \leq i \leq K$, for each ϵ, we must restrict the operator

$$A_\epsilon u_\epsilon = -\sum_{i,j=1}^{N} D_i(a^{ij}_\epsilon D_j u_\epsilon) + \sum_{i=1}^{K} b^i u_\epsilon^{n_i},$$

to a subset of functions in $C^\infty(\overline{\Omega})$ to guarantee that A_ϵ is well-defined. In particular, for each ϵ we consider functions $u_\epsilon \in C^\infty(\overline{\Omega})$ such that $0 < \alpha_\epsilon \leq u_\epsilon \leq \beta_\epsilon < \infty$ for some choice of α_ϵ and β_ϵ. The first part of this section is dedicated to making judicious choices of α_ϵ and β_ϵ for each ϵ such that a solution u_ϵ to (4.19) exists that satisfies $\alpha_\epsilon \leq u_\epsilon \leq \beta_\epsilon$.

Once a net of solutions (u_ϵ) is determined, it is necessary to show that if $(u_\epsilon) \in \mathcal{E}_M(\overline{\Omega})$, then an operator $A \in \mathcal{A}_0$ whose coefficients satisfy (4.1) is well-defined for (u_ϵ). Recall that A is only a well defined operator for elements $u \in \mathcal{G}(\overline{\Omega})$ satisfying $u_\epsilon \geq C\epsilon^a$ for $\epsilon \in (0, \epsilon_0) \subset (0, 1)$, $a \in \mathbb{R}$ and some constant C independent of ϵ. This will require us to establish certain ϵ-decay estimates on α_ϵ, which we do later in this section.

5.1. L^∞ **Bounds for the Semilinear Problem.** We begin by determining the net of *a priori* bounds α_ϵ and β_ϵ described above. For now we disregard the ϵ notation. In the following proposition we determine *a priori* estimates for a positive solution u to a

problem of the form

$$-\sum_{i,j}^{N} D_i(a^{ij}D_j u) + \sum_{i=1}^{K} b^i u^{n_i} = 0 \text{ in } \Omega, \tag{5.1}$$
$$u|_{\partial\Omega} = \rho,$$

where $\Omega \subset \mathbb{R}^n$ is connected, bounded, and of C^∞-class, and $a^{ij}, b^i, \rho \in C(\overline{\Omega})$ with $\rho > 0$ in $\overline{\Omega}$.

Proposition 5.1. *Suppose that the semilinear operator in* (5.1) *has the property that* $n_i > 0$ *for some* $1 \leq i \leq K$. *Let* n_K *be the largest positive exponent and suppose that* $b_K(x) > 0$ *in* $\overline{\Omega}$. *Additionally, assume that one of the following two cases holds:*

$$\text{(1) } n_i < 0 \text{ for some } 1 \leq i \leq K \text{ and if } n_1 = \min\{n_i : n_i < 0\}, \tag{5.2}$$
$$\text{then } b_1(x) < 0 \text{ in } \overline{\Omega}.$$
$$\text{(2) } n_K \text{ is odd and } 0 < n_i \text{ for all } 1 \leq i \leq K. \tag{5.3}$$

Then if case (5.2) *holds and* u *is a nonnegative solution to* (5.1), *there exist constants* α *and* β *such that* $0 < \alpha \leq u \leq \beta$. *If case* (5.3) *holds and* u *is a nonnegative solution to* (5.1), *there exists a constant* β *such that* $0 \leq u \leq \beta$.

Remark 5.2. *Note that for the purposes of proving Theorem 4.1, we are primarily concerned with case* (5.2). *This is the case that we will focus on for the remainder of the paper. However, with a little extra work we could very easily generalize Theorem 4.1 to allow for* $n_i > 0$ *for all* $1 \leq i \leq K$ *and* $n_K > 0$ *odd. Then we could use case* (5.3) *to establish the necessary bounds.*

Proof. We first define α' and β' in the following way. If case (5.2) holds, then let

$$\alpha' = \sup_{c\in\mathbb{R}_+} \left\{ \sum_{i=1}^{K} \sup_{x\in\Omega} b_i(x) b^{n_i} < 0 \quad \forall b \in (0, c) \right\}. \tag{5.4}$$

If case (5.3) holds, let $\alpha' = 0$. In either case (5.2) or (5.3), define

$$\beta' = \inf_{c\in\mathbb{R}} \left\{ \sum_{i=1}^{K} \inf_{x\in\Omega} b_i(x) b^{n_i} > 0 \quad \forall b \in (c, \infty) \right\}. \tag{5.5}$$

Note that in all cases α' and β' are always well-defined given the conditions on $b^1(x)$ and $b^K(x)$. Then define

$$\alpha = \min\{\alpha', \inf_{x\in\partial\Omega} \rho(x)\}, \tag{5.6}$$

$$\beta = \max\{\beta', \sup_{x\in\partial\Omega} \rho(x)\}. \tag{5.7}$$

Based on these definitions of α and β, if u is a nonnegative solution to (5.1) then it is easy to verify that the functions $\overline{\phi} = (u - \beta)^+$ and $\underline{\phi} = (u - \alpha)^-$ are in $H_0^1(\Omega)$ if either (5.2) or (5.3) holds. Define the set

$$\overline{\mathcal{Y}} = \left\{ x \in \overline{\Omega} \mid u \geq \beta \right\}$$

if case (5.2) or (5.3) holds. If case (5.2) holds, let

$$\underline{\mathcal{Y}} = \{x \in \overline{\Omega} \mid 0 < u \leq \alpha\},$$

if case (5.3) holds, let

$$\underline{\mathcal{Y}} = \{x \in \overline{\Omega} \mid u < \alpha\}.$$

Then if $u \in H^1(\Omega)^+$ is a weak solution to (5.1), supp($\overline{\phi}$) = $\overline{\mathcal{Y}}$ and supp($\underline{\phi}$) = $\underline{\mathcal{Y}}$. We have the following string of inequalities for $\underline{\phi}$:

$$\begin{aligned} C_2\|\underline{\phi}\|^2_{H^1(\Omega)} &\leq C_1\|\nabla((u-\alpha)^-)\|^2_{L^2(\Omega)} \\ &\leq \int_\Omega a^{ij}D_j((u-\alpha)^-)D_j((u-\alpha)^-)\,dx \\ &= \int_\Omega a^{ij}D_j(u-\alpha)D_j((u-\alpha)^-)\,dx \\ &= \int_{\underline{\mathcal{Y}}}(-\sum_{i=1}^K b_i(x)u^{n_i})(u-\alpha)\,dx \leq 0. \end{aligned} \tag{5.8}$$

Similarly, we have the following string of inequalities for $\overline{\phi}$:

$$\begin{aligned} C_2\|\overline{\phi}\|^2_{H^1(\Omega)} &\leq C_1\|\nabla((u-\beta)^+)\|^2_{L^2(\Omega)} \\ &\leq \int_\Omega a^{ij}D_j((u-\beta)^+)D_i((u-\beta)^+)\,dx \\ &= \int_\Omega a^{ij}D_j(u-\beta)D_i((u-\beta)^+)\,dx \\ &= \int_{\overline{\mathcal{Y}}}(-\sum_{i=1}^K b_i(x)u^{n_i})(u-\beta)\,dx \leq 0. \end{aligned} \tag{5.9}$$

The above inequalities imply that if u is a positive, weak solution to the semilinear (5.1), then $u \in [\alpha, \beta]$ where in case (5.2), $\alpha > 0$. □

Now that we've established L^∞-bounds for solutions to (5.1), we can apply these bounds for each fixed ϵ to determine a net of bounds for the following net of problems:

$$\begin{aligned} -\sum_{i,j}^N D_i a^{ij}_\epsilon D_j u_\epsilon + \sum_{i=1}^K b^i_\epsilon u_\epsilon^{n_i} = 0 \quad &\text{in } \Omega \\ u_\epsilon|_{\partial\Omega} = \rho_\epsilon, & \end{aligned} \tag{5.10}$$

where $(a^{ij}_\epsilon), (b^i_\epsilon), (\rho_\epsilon) \in \mathcal{E}_M(\overline{\Omega})$ satisfy the following for all $\epsilon < 1$:

$$\begin{aligned} & a^{ij}_\epsilon = a^{ji}_\epsilon, \quad a^{ij}_\epsilon \xi_i\xi_j \geq \lambda_\epsilon|\xi|^2 \geq C_1\epsilon^{a_1}|\xi|^2 \\ & |a^{ij}_\epsilon|_{k,\alpha;\Omega}, \ |b^i_\epsilon|_{k,\alpha;\Omega} \leq \Lambda_{k,\epsilon} \leq C_2\epsilon^{a_2}, \quad \forall k \in \mathbb{N} \\ & b^1_\epsilon \leq -C_3\epsilon^{a_3}, \quad \{n_i : n_i < 0\} \neq \emptyset, \quad n_1 = \min\{n_i : n_i < 0\} \\ & b^K_\epsilon \geq C_4\epsilon^{a_4}, \quad \{n_i : n_i > 0\} \neq \emptyset, \quad n_K = \max\{n_i : n_i > 0\} \\ & \rho_\epsilon \geq C_5\epsilon^{a_5}, \end{aligned} \tag{5.11}$$

and $C_1, \cdots, C_5$ are independent of ϵ and $a_1, \cdots, a_5 \in \mathbb{R}$.

Proposition 5.3. *Suppose that for each fixed $\epsilon \in (0,1]$, u_ϵ is a positive solution to* (5.10) *with coefficients satisfying* (5.11)*. Then there exist L^∞-bounds α_ϵ and β_ϵ such that for each ϵ, $0 < \alpha_\epsilon \leq u_\epsilon \leq \beta_\epsilon$.*

Proof. For each fixed ϵ, if the assumptions in (5.11) hold, then case (5.2) of Proposition 5.1 is satisfied. Therefore, for each $\epsilon \in (0,1]$, there exists α_ϵ and β_ϵ such that $0 < \alpha_\epsilon \leq u_\epsilon \leq \beta_\epsilon$. □

5.2. **Sub- and Super-Solutions.** In the previous section we showed that if the data of (5.10) satisfies (5.11) and if $u_\epsilon \in C^\infty(\overline{\Omega})$ solves (5.10) for each ϵ, then $0 < \alpha_\epsilon \leq u_\epsilon \leq \beta_\epsilon$. Now, for each $\epsilon \in (0,1]$, we want to show that there actually exists a solution $u_\epsilon \in C^\infty(\overline{\Omega})$ satisfying $0 < \alpha_\epsilon \leq u_\epsilon \leq \beta_\epsilon$. The key to proving this result lies in the fact that α_ϵ and β_ϵ are sub- and super-solutions to (5.10) for each ϵ.

Proposition 5.4. *Let α_ϵ and β_ϵ be the bounds established in Proposition 5.3. Suppose that the coefficients in the net of problems* (5.10) *satisfy* (5.11) *and that for each ϵ some $b^i_\epsilon(x)$ is non-constant. Then there exists a net $(u_\epsilon) \in (C(\overline{\Omega}))^I$ such that for each ϵ, u_ϵ solves* (5.10) *and $0 < \alpha_\epsilon \leq u_\epsilon \leq \beta_\epsilon$.*

Proof. To solve the above family of problems in (5.10), we show that the net of L^∞-bounds (α_ϵ) and (β_ϵ) found in Proposition 5.3 is a net of sub and super-solutions to (5.10). Then we verify that the interval $[\alpha_\epsilon, \beta_\epsilon]$ is nonempty and is a subset of $\mathbb{R}^+$. We will then be able to apply Theorem 4.3 to conclude that for each ϵ, there exists a solution $u_\epsilon \in C^\infty(\overline{\Omega})$.

Fix ϵ and let α'_ϵ and β'_ϵ be defined by (5.4) and (5.5) respectively, and let

$$\begin{aligned} \alpha_\epsilon &= \min\{\alpha'_\epsilon, \inf_{\partial\Omega} \rho_\epsilon(x)\}, \\ \beta_\epsilon &= \max\{\beta'_\epsilon, \sup_{x\in\partial\Omega} \rho_\epsilon(x)\}. \end{aligned}$$

By (5.4) and the fact that $\rho_\epsilon > 0$, we clearly have that $\alpha_\epsilon > 0$. Then (5.4) and the definition of α_ϵ imply that

$$\begin{aligned} A_\epsilon \alpha_\epsilon = \sum_{i=1}^K b^i_\epsilon(\alpha_\epsilon)^{n_i} \leq \sum_{i=1}^K \sup_{x\in\overline{\Omega}} b^i_\epsilon(\alpha_\epsilon)^{n_i} \leq 0, \\ \alpha_\epsilon \leq \inf_{x\in\partial\Omega} \rho_\epsilon(x) \leq \rho_\epsilon, \end{aligned} \tag{5.12}$$

which shows that α_ϵ is sub-solution for each ϵ. Similarly, (5.5) and the definition of β'_ϵ imply that

$$\begin{aligned} A_\epsilon \beta_\epsilon = \sum_{i=1}^K b^i_\epsilon(\beta_\epsilon)^{n_i} \geq \sum_{i=1}^K \inf_{x\in\overline{\Omega}} b^i_\epsilon(\beta_\epsilon)^{n_i} \geq 0, \\ \beta_\epsilon \geq \sup_{x\in\partial\Omega} \rho_\epsilon \geq \rho_\epsilon, \end{aligned} \tag{5.13}$$

which shows that β_ϵ is a super-solution for each ϵ.

Now that we've determined that the pair α_ϵ and β_ϵ are sub- and super-solutions of (5.10), we show that the interval $[\alpha_\epsilon, \beta_\epsilon]$ is nonempty. Given the definition of α_ϵ and β_ϵ, it suffices to show that $\alpha'_\epsilon < \beta'_\epsilon$. We claim that $\alpha'_\epsilon < \beta'_\epsilon$ provided that at least one $b^i_\epsilon(x)$ is nonzero and non-constant. This follows from the fact that if we define

$$\gamma_\epsilon = \inf_{c\in\mathbb{R}} \{\sum_{i=1}^K \sup_{x\in\overline{\Omega}} b^i(x) d^{n_i} \geq 0 \quad \forall d \in (c, \infty)\},$$

then we have that $\alpha'_\epsilon \leq \gamma_\epsilon$ by (5.4) and (5.11). Furthermore, for a fixed ϵ, given the assumptions on $b^i_\epsilon(x)$,

$$\sum_{i=1}^K \inf_{x\in\overline{\Omega}} b^i_\epsilon(x) y^{n_i} < \sum_{i=1}^K \sup_{x\in\overline{\Omega}} b^i_\epsilon(x) y^{n_i} \quad \forall y \in \mathbb{R}.$$

But (5.5) and the above inequality clearly imply that $\gamma_\epsilon < \beta'_\epsilon$. Therefore $\alpha'_\epsilon < \beta'_\epsilon$ and the interval $[\alpha_\epsilon, \beta_\epsilon]$ is a nonempty subset of $\mathbb{R}^+$. For each $\epsilon \in (0,1]$, the hypotheses of

Theorem 4.3 are satisfied for the elliptic problem (5.11), so we may conclude that there exists a net of solutions $(u_\epsilon) \in (C^\infty(\overline{\Omega}))^I$ that satisfy $0 < \alpha_\epsilon \le u_\epsilon \le \beta_\epsilon$ for each fixed ϵ. □

The final task in this section is to show that an operator $A \in \mathcal{A}_0$, with coefficients satisfying (5.11), is a well-defined operator on any element $u \in \mathcal{E}_M(\overline{\Omega})$ satisfying

$$\alpha_\epsilon \le u_\epsilon \le \beta_\epsilon \quad \forall \epsilon \in (0,1].$$

Recall that in Section 3.4 we determined that A is only well-defined for invertible $u \in \mathcal{G}(\overline{\Omega})$. Therefore, it suffices to show that $(\alpha_\epsilon), (\beta_\epsilon)$ and $(\frac{1}{\alpha_\epsilon}), (\frac{1}{\beta_\epsilon})$ are generalized constants (3.3), which we verify in the following lemma.

Lemma 5.5. *Let (α_ϵ) and (β_ϵ) be the net of sub- and super-solutions to* (5.10) *determined in Section 5.1. Suppose that the coefficients of* (5.10) *satisfy* (5.11)*. Then (α_ϵ), (β_ϵ),$(\frac{1}{\alpha_\epsilon})$, and $(\frac{1}{\beta_\epsilon})$ are in $\overline{\mathbb{C}}$, the ring of generalized constants.*

Remark 5.6. *Note that if $(\frac{1}{\alpha_\epsilon}) \in \overline{\mathbb{C}}$, then this implies that there exists an $\epsilon_0 \in (0,1)$, some constant C independent of ϵ and $a \in \mathbb{R}$ such that $\alpha_\epsilon \ge C\epsilon^a$ for all $\epsilon \in (0,\epsilon_0)$. Then if $(u_\epsilon) \in \mathcal{E}_M(\overline{\Omega})$ satisfies $\alpha_\epsilon \le u_\epsilon \le \beta_\epsilon$ for each ϵ, $(\frac{1}{\alpha_\epsilon}) \in \overline{\mathbb{C}}$ implies that $u = [(u_\epsilon)]$ is invertible in $\mathcal{G}(\overline{\Omega})$. See Section 3.4 and* [6] *for more details.*

Proof. We need to show that there exists constants D_1, D_2 independent of ϵ and $\epsilon_0 \in (0,1)$ such that for all $\epsilon \in (0,\epsilon_0)$,

$$\alpha_\epsilon \ge D_1\epsilon^{b_1} \quad \text{for some } b_1 \in \mathbb{R},$$
$$\beta_\epsilon \le D_2\epsilon^{b_2} \quad \text{for some } b_2 \in \mathbb{R}.$$

So it is necessary to verify that there exists constants D_1 and D_2 so that for ϵ sufficiently small

$$\alpha'_\epsilon \ge D_1\epsilon^{b_1}, \quad \text{and} \quad \inf_{x\in\partial\Omega} \rho_\epsilon \ge D_1\epsilon^{b_1},$$
$$\beta'_\epsilon \le D_2\epsilon^{b_2}, \quad \text{and} \quad \sup_{x\in\partial\Omega} \rho_\epsilon \le D_2\epsilon^{b_2}.$$

Given that $(\rho_\epsilon) \in \mathcal{E}_M(\overline{\Omega})$,

$$\sup_{x\in\partial\Omega} \rho_\epsilon \le \sup_{x\in\overline{\Omega}} \rho_\epsilon = \mathcal{O}(\epsilon^b),$$

for some $b \in \mathbb{R}$. This and the assumption on (ρ_ϵ) in (5.11) imply that we only need to obtain the necessary ϵ-bounds on α'_ϵ and β'_ϵ.

For now, drop the ϵ notation and consider α' defined in (5.4). For a given function f, define

$$\underline{\gamma}_f = \sup_{c\in\mathbb{R}_+} \{f(b) \le 0 \quad \forall b \in (0,c)\}.$$

Given that

$$\alpha' = \sup_{c\in\mathbb{R}_+} \left\{ \sum_{i=1}^{K} \sup_{x\in\Omega} b_i(x) b^{n_i} \le 0 \quad \forall b \in (0,c) \right\},$$

it is clear that for another function $f(y)$ such that

$$f(y) \ge \sum_{i=1}^{K} \sup_{x\in\Omega} b_i(x) y^{n_i} \quad \text{on } (0,c),$$

if $\underline{\gamma}_f$ is defined and $\alpha' \in (0,c)$, it must hold that $\underline{\gamma}_f \le \alpha'$. Let $C_1 = |\{n_i : n_i \ge 0\}|$ and $C_2 = |\{n_i : n_i < 0\}|$ and if $C_2 > 1$, let $n_{i_2} = \min\{n_i : n_1 < n_i < 0\}$. Note that

$C_1, C_2 \geq 1$ based on the assumptions in (5.11). Then recalling that $b_1(x) < 0$, $b_K(x) > 0$ correspond to the coefficients of the terms with the smallest negative and largest positive exponent of $\sum_i^K b_i(x)u^{n_i}$, if $\sup_{x\in\overline{\Omega}} |b_i(x)| \leq \Lambda$ for each i, the following must hold for $y \in (0,1)$:

$$\sum_{i=1}^{K} \sup_{x\in\Omega} b_i(x) y^{n_i} \leq \sup_{x\in\overline{\Omega}} b_1(x) y^{n_1} + C_1\Lambda + (C_2-1)\Lambda y^{n_{i_2}}. \tag{5.14}$$

Define

$$d = \left(\frac{-\sup_{x\in\overline{\Omega}}(b_1(x))}{2(C_2-1)\Lambda} \right)^{\frac{1}{n_{i_2}-n_1}}$$

if $C_2 > 1$ and let $d = 1$ if $C_2 = 1$. Then let $c = \min\{1, d\}$. The definition of c implies that

$$(C_2-1)\Lambda y^{n_{i_2}} \leq -\frac{\sup_{x\in\overline{\Omega}} b_1(x)}{2} y^{n_1},$$

for all $y \in (0,c)$. So for $y \in (0,c)$,

$$\sum_{i=1}^{K} \sup_{x\in\overline{\Omega}} b_i(x) y^{n_i} \leq \frac{\sup_{x\in\overline{\Omega}} b_1(x)}{2} y^{n_1} + C_1\Lambda = f(y).$$

Then if $\alpha' \in (0,c)$, $\alpha' \geq \underline{\gamma}_f$. Given that $f(y)$ is a monotone increasing function on $\mathbb{R}_+$, $\underline{\gamma}_f$ is the lone positive root of $f(y)$. Thus,

$$\underline{\gamma}_f = \left(\frac{-\sup_{x\in\overline{\Omega}} b_1(x)}{2C_1\Lambda} \right)^{\frac{1}{-n_1}},$$

which implies that if $\alpha' \in (0,c)$,

$$\alpha' \geq \left(\frac{-\sup_{x\in\Omega} b_1(x)}{2C_1\Lambda} \right)^{\frac{1}{-n_1}}.$$

Similarly, for a fixed $\epsilon \in (0,1)$, define

$$d_\epsilon = \left(\frac{-\sup_{x\in\overline{\Omega}}(b_\epsilon^1(x))}{2(C_2-1)\Lambda_\epsilon} \right)^{\frac{1}{n_{i_2}-n_1}},$$

if $C_2 > 1$ and let $d_\epsilon = 1$ if $C_2 = 1$. Let $c_\epsilon = \min\{1, d_\epsilon\}$. Then for $y \in (0, c_\epsilon)$, we have that

$$(C_2-1)\Lambda_\epsilon y^{n_{i_2}} \leq -\frac{\sup_{x\in\overline{\Omega}} b_\epsilon^1(x)}{2} y^{n_1}.$$

So the above arguments imply that if $\alpha'_\epsilon \in (0, c_\epsilon)$, then $\alpha'_\epsilon \geq \underline{\gamma}_{f,\epsilon}$ and

$$\alpha'_\epsilon \geq \left(\frac{-\sup_{x\in\overline{\Omega}} b_\epsilon^1(x)}{2C_1\Lambda_\epsilon} \right)^{\frac{1}{-n_1}}.$$

Given the assumptions on $b_\epsilon^1(x)$ and Λ_ϵ in (5.11), in this case we have that $\alpha'_\epsilon \geq C\epsilon^a$ for some constant $C > 0$, $a \in \mathbb{R}$ and ϵ sufficiently small. Now we must show that $c_\epsilon \geq C\epsilon^a$ for some constant $C > 0$, $a \in \mathbb{R}$ and ϵ sufficiently small in the event that $\alpha'_\epsilon \notin (0, c_\epsilon)$. It suffices to show that $d_\epsilon \geq C\epsilon^a$ in the event that $C_2 > 1$. But clearly, for ϵ sufficiently small

$$d_\epsilon = \left(-\frac{\sup_{x\in\overline{\Omega}} b_\epsilon^1(x)}{2(C_2-1)\Lambda_\epsilon} \right)^{\frac{1}{n_{i_2}-n_1}} \geq C\epsilon^a,$$

given the assumptions on b^1_ϵ and Λ_ϵ in (5.11). Therefore $\alpha'_\epsilon \geq D_1\epsilon^a$ for some constant $D_1 > 0$, $a \in \mathbb{R}$ and ϵ sufficiently small.

Now we determine bounds on the net (β'_ϵ). Again, we temporarily drop the ϵ and only consider β'. Recall that

$$\beta' = \inf_{c\in\mathbb{R}} \left\{ \sum_{i=1}^{K} \inf_{x\in\Omega} b_i(x) b^{n_i} \geq 0 \quad \forall b \in (c,\infty) \right\}.$$

For a given function $f(y)$, define

$$\overline{\gamma}_f = \inf_{c\in\mathbb{R}} \left\{ f(b) \geq 0 \quad \forall b \in (c,\infty) \right\}.$$

Then if $f(y) \leq \sum_{i=1}^K \sup_{x\in\Omega} b_i(x) y^{n_i}$ on some interval (c,∞) and $\beta' \in (c,\infty)$, it must hold that $\overline{\gamma}_f \geq \beta'$ if $\overline{\gamma}_f$ is defined. Let C_1, C_2 be as before and let $n_{i_1} = \max\{n_i : 0 \leq n_i < n_K\}$ if $C_1 > 1$. If $y > 1$, then

$$\sum_{i=1}^{K} \inf_{x\in\Omega} b_i(x) y^{n_i} \geq \inf_{x\in\overline{\Omega}} (b_K(x)) y^{n_K} - (C_1 - 1)\Lambda y^{n_{i_1}} - C_2\Lambda.$$

Now define

$$d = \left(\frac{2(C_1-1)\Lambda}{\inf_{x\in\overline{\Omega}}(b_K(x))} \right)^{\frac{1}{n_k - n_{i_1}}}$$

if $C_1 > 1$ and let $d = 1$ if $C_1 = 1$. Let $c = \max\{1, d\}$. Then our choice of d ensures that if $C_1 > 1$, then

$$-(C_1-1)\Lambda y^{n_{i_1}} \geq -\frac{\inf_{x\in\overline{\Omega}}(b_K(x)) y^{n_K}}{2},$$

and that for $y \in (c,\infty)$,

$$\sum_{i=1}^{K} \sup_{x\in\Omega} b_i(x) y^{n_i} \geq \frac{\inf_{x\in\overline{\Omega}}(b_K(x))}{2} y^{n_K} - C_2\Lambda = f(y).$$

So if $\beta' \in (c,\infty)$, $\beta' \leq \overline{\gamma}_f$, where $\overline{\gamma}_f$ is the lone positive root of f on $\mathbb{R}_+$ given that f is monotone increasing on this interval. So if $\beta' \in (c,\infty)$,

$$\beta' \leq \overline{\gamma}_f = \left(\frac{2C_2\Lambda}{\inf_{x\in\overline{\Omega}}(b_K(x))} \right)^{\frac{1}{n_K}}.$$

By defining

$$d_\epsilon = \left(\frac{2(C_1-1)\Lambda_\epsilon}{\inf b^K_\epsilon(x)} \right)^{\frac{1}{n_k - n_{i_1}}} \quad \text{and} \quad c_\epsilon = \max\{1, d_\epsilon\} \tag{5.15}$$

and applying the above argument for β' to the net (β'_ϵ) for each fixed ϵ, it is clear that if $\beta'_\epsilon \in (c_\epsilon, \infty)$, then

$$\beta'_\epsilon \leq \left(\frac{2C_2\Lambda_\epsilon}{\inf b^K_\epsilon(x)} \right)^{\frac{1}{n_K}} \leq C\epsilon^a$$

given the assumptions on b^K_ϵ and Λ_ϵ in (5.11).

Now assume that $\beta'_\epsilon \notin (c_\epsilon, \infty)$. Then it suffices to show that if $C_1 > 1$, then $d_\epsilon \leq C\epsilon^a$ for ϵ sufficiently small, some positive constant C and $a \in \mathbb{R}$. But again, this is clearly true given the assumptions (5.11) and the fact that

$$d_\epsilon = \left(\frac{2(C_1-1)\Lambda_\epsilon}{\inf b^K_\epsilon(x)} \right)^{\frac{1}{n_k - n_{i_1}}}.$$

□

6. Proof of the Main Results

We now prove Theorem 4.1 using the results from Section 5. For clarity, we break the proof up into the steps outlined in Section 4.2.

6.1. **Proof of Theorem 4.1.**

Proof. Step 1: *Formulation of the problem.* For convenience, we restate the problem and the formulation that we will use to find a solution. Given an operator $A \in \mathcal{A}_0$, defined by (3.15), we want to solve the following Dirichlet problem in $\mathcal{G}(\overline{\Omega})$:

$$Au = 0 \quad \text{in } \Omega, \tag{6.1}$$

$$u|_{\partial\Omega} = \rho \quad . \tag{6.2}$$

We phrase (6.1) in a way that allows us to solve a net of semilinear elliptic problems. We assume that the coefficients of A and boundary data ρ have representatives $(a_\epsilon^{ij}), (b_\epsilon^i)$, and (ρ_ϵ) in $\mathcal{E}_M(\overline{\Omega})$ satisfying the assumptions (4.1). Then for this particular choice of representatives, our strategy for solving (6.1) is to solve the family of problems

$$A_\epsilon u_\epsilon = -\sum_{i,j=1}^{N} D_i(a_\epsilon^{ij} D_j u_\epsilon) + \sum_{i}^{N} b_\epsilon^i u_\epsilon^{n_i} = 0 \text{ in } \Omega, \tag{6.3}$$

$$u_\epsilon|_{\partial\Omega} = \rho_\epsilon, \tag{6.4}$$

and then show that the net of solutions $(u_\epsilon) \in \mathcal{E}_M(\overline{\Omega})$.

Step 2: *Determine L^∞-estimates and a net of sub-solutions and super-solutions*. In Section 5, we concluded that for each ϵ, the pair α_ϵ and β_ϵ determine sub- and super-solutions to (6.3) such that $0 < \alpha_\epsilon < \beta_\epsilon$. Furthermore, in Lemma 5.5 we concluded that there exist $C_1, C_2 > 0$ and $a_1, a_2 \in \mathbb{R}$ such that for ϵ sufficiently small, the nets (α_ϵ) and (β_ϵ) satisfy $C_1\epsilon^{a_1} \le \alpha_\epsilon < \beta_\epsilon \le C_2\epsilon^{a_2}$, thereby verifying that $(\alpha_\epsilon), (\beta_\epsilon), (\frac{1}{\alpha_\epsilon}), (\frac{1}{\beta_\epsilon}) \in \overline{\mathbb{C}}$, the ring of generalized constants.

Step 3: *Apply fixed point theorem to solve each semilinear problem in* (4.19). This follows from Proposition 5.4. We briefly reiterate the proof here. We simply verify the hypotheses of Theorem 4.3. For each fixed ϵ we have sub- and super-solutions α_ϵ and β_ϵ satisfying $0 < \alpha_\epsilon < \beta_\epsilon$ and $a_\epsilon^{ij}, b_\epsilon^i, \rho_\epsilon \in C^\infty(\overline{\Omega})$ satisfying (5.11). Finally, Ω is of C^∞-class and the function

$$f(x,y) = -\sum_{i=1}^{K} b_\epsilon^i(x) y^{n_i} \in C^\infty(\overline{\Omega} \times \mathbb{R}^+),$$

so we may apply Theorem 4.3 to conclude that there exists a net of solutions (u_ϵ) to (5.10) satisfying $0 < \alpha_\epsilon \le u_\epsilon \le \beta_\epsilon$.

Step 4: *Verify that the net of solutions* $(u_\epsilon) \in \mathcal{E}_M(\overline{\Omega})$. Now that it is clear that a solution exists for (5.10) for each $\epsilon \in (0,1]$, it is necessary to establish estimates that show that the net of solutions (u_ϵ) is in $\mathcal{E}_M(\overline{\Omega})$. That is, we want to show that for each $k \in \mathbb{N}$ and all multi-indices $|\beta| \le k$, there exists $a \in \mathbb{R}$ such that

$$\sup_{x \in \overline{\Omega}} \{|D^\beta u_\epsilon(x)|\} = \mathcal{O}(\epsilon^a).$$

By standard interpolation inequalities, it suffices to show that for $\gamma \in (0,1)$ and each $k \in \mathbb{N}$, there exists an $a \in \mathbb{R}$ such that

$$|u_\epsilon|_{k,\gamma;\Omega} = \mathcal{O}(\epsilon^a).$$

By Theorem 3.1, we have that if u_ϵ is a solution to (5.10) with coefficients satisfying (5.11), then

$$|u_\epsilon|_{2,\gamma;\Omega} \leq C\left(\frac{\Lambda_\epsilon}{\lambda_\epsilon}\right)^3 (|u_\epsilon|_{0;\Omega} + |\rho_\epsilon|_{2,\gamma;\Omega} + \sum_{i=1}^K |b_\epsilon^i(u_\epsilon)^{n_i}|_{0,\gamma;\Omega}). \tag{6.5}$$

Observe that

$$|u_\epsilon^{n_i}|_{0,\gamma;\Omega} \leq |u_\epsilon^{n_i}|_{0;\Omega} + n_i[u_\epsilon]_{0,\gamma;\Omega}|u_\epsilon|_{0;\Omega}^{n_i-1} \tag{6.6}$$

if $n_i > 0$ and

$$|u_\epsilon^{n_i}|_{0,\gamma;\Omega} \leq |u_\epsilon^{n_i}|_{0;\Omega} + \frac{1}{|u_\epsilon^{-n_i}|_{0;\Omega}^2}(-n_i)[u_\epsilon]_{0,\gamma;\Omega}|u_\epsilon|_{0;\Omega}^{-n_i-1}, \tag{6.7}$$

if $n_i < 0$. The above inequality implies that

$$\begin{aligned} |u_\epsilon|_{2,\gamma;\Omega} \leq C\left(\frac{\Lambda_\epsilon}{\lambda_\epsilon}\right)^3 (&|u_\epsilon|_{0;\Omega} + |\rho_\epsilon|_{2,\gamma;\Omega} \\ &+ \sum_{i=1}^K |b_\epsilon^i(x)|_{0,\gamma;\Omega}(C_1(\alpha_\epsilon,\beta_\epsilon,n_i) + C_2(n_i,\alpha_\epsilon,\beta_\epsilon)|u_\epsilon|_{0,\gamma;\Omega})), \end{aligned} \tag{6.8}$$

where

$$C_1(n_i,\alpha_\epsilon,\beta_\epsilon) = \beta_\epsilon^{n_i} \quad \text{and} \quad C_2(n_i,\alpha_\epsilon,\beta_\epsilon) = n_i\beta_\epsilon^{n_i-1}, \quad \text{if} \quad n_i > 0 \quad \text{and}$$

$$C_1(n_i,\alpha_\epsilon,\beta_\epsilon) = \alpha_\epsilon^{n_i} \quad \text{and} \quad C_2(n_i,\alpha_\epsilon,\beta_\epsilon) = \frac{(-n_i)\beta_\epsilon^{-n_i-1}}{\alpha_\epsilon^{-2n_i}} \quad \text{if} \quad n_i < 0.$$

Application of the interpolation inequality

$$|u_\epsilon|_{0,\gamma} \leq C(\delta_\epsilon^{-1}|u_\epsilon|_0 + \delta_\epsilon|u_\epsilon|_{2,\gamma}),$$

where δ_ϵ is arbitrarily small and C is independent of δ_ϵ, implies that

$$\begin{aligned} |u_\epsilon|_{2,\gamma;\Omega} \leq C\left(\frac{\Lambda_\epsilon}{\lambda_\epsilon}\right)^3 (&|u_\epsilon|_{0;\Omega} + |\rho_\epsilon|_{2,\gamma;\Omega} \\ &+ \sum_{i=1}^K |b_\epsilon^i(x)|_{0,\gamma;\Omega}(C_1(n_i,\alpha_\epsilon,\beta_\epsilon) \\ &+ C_2(n_i,\alpha_\epsilon,\beta_\epsilon)(C(\delta_\epsilon^{-1}|u_\epsilon|_{0;\Omega} + \delta_\epsilon|u_\epsilon|_{2,\gamma;\Omega})))). \end{aligned} \tag{6.9}$$

Therefore,

$$\begin{aligned} &\left(1 - \delta_\epsilon\left(\frac{\Lambda_\epsilon}{\lambda_\epsilon}\right)\sum_{i=1}^K |b_\epsilon^i(x)|_{0,\gamma;\Omega}C_2(n_i,\alpha_\epsilon,\beta_\epsilon)\right)|u_\epsilon|_{2,\gamma;\Omega} \\ &\qquad \leq C\left(\frac{\Lambda_\epsilon}{\lambda_\epsilon}\right)^3 (|u_\epsilon|_{0;\Omega} + |\rho_\epsilon|_{2,\gamma;\Omega} \\ &\qquad\quad + \sum_{i=1}^K |b_\epsilon^i(x)|_{0,\gamma;\Omega}(C_1(n_i,\alpha_\epsilon,\beta_\epsilon) + C_2(n_i,\alpha_\epsilon,\beta_\epsilon)\delta_\epsilon^{-1}|u_\epsilon|_{0;\Omega})). \end{aligned} \tag{6.10}$$

But given the assumptions on Λ_ϵ, λ_ϵ, the bounds previously established for the nets (α_ϵ) and (β_ϵ) in Lemma 5.5, and given that $(b^i_\epsilon(x)) \in \mathcal{E}_M(\overline{\Omega})$, there exists $\epsilon_0 \in (0,1)$, $a \in \mathbb{R}$ and $C > 0$ such that for all $\epsilon \in (0,\epsilon_0)$,

$$\left(\frac{\Lambda_\epsilon}{\lambda_\epsilon}\right) \sum_{i=1}^{K} |b^i_\epsilon(x)|_{0,\gamma} C_2(n_i,\alpha_\epsilon,\beta_\epsilon) \leq C\epsilon^a.$$

Therefore, choosing

$$\delta_\epsilon = \frac{1}{2C\epsilon^a},$$

it is clear that for $\epsilon \in (0,\epsilon_0)$,

$$\begin{aligned} |u_\epsilon|_{2,\gamma;\Omega} \leq C \left(\frac{\Lambda_\epsilon}{\lambda_\epsilon}\right)^3 (|u_\epsilon|_{0;\Omega} + |\rho_\epsilon|_{2,\gamma;\Omega} \\ + \sum_{i=1}^{K} |b^i_\epsilon(x)|_{0,\gamma;\Omega} (C_1(n_i,\alpha_\epsilon,\beta_\epsilon) + C_2(n_i,\alpha_\epsilon,\beta_\epsilon,\epsilon^a)|u_\epsilon|_{0;\Omega})). \end{aligned} \tag{6.11}$$

Given that (α_ϵ), $(\beta_\epsilon) \in \overline{\mathbb{C}}$, $\alpha_\epsilon \leq u_\epsilon \leq \beta_\epsilon$ and (ρ_ϵ), $(b^i_\epsilon) \in \mathcal{E}_M(\overline{\Omega})$, the above inequality implies that for some $a \in \mathbb{R}$,

$$|u_\epsilon|_{2,\gamma;\Omega} = \mathcal{O}(\epsilon^a).$$

Now we need to utilize the ϵ-growth conditions on $|u_\epsilon|_{2,\gamma;\Omega}$ and induction to show that for any $k > 2$ that

$$|u_\epsilon|_{k,\gamma;\Omega} = \mathcal{O}(\epsilon^a) \quad \text{for some } a \in \mathbb{R}. \tag{6.12}$$

Let (u_ϵ) be a smooth net of solutions to (6.3) and additionally assume that (6.12) holds for all $j \leq k$. Let ν be a multi-index of length $k-1$. Then by differentiating both sides of (6.3), we see that for each ϵ, u_ϵ satisfies the Dirichlet problem

$$\begin{aligned} \sum_{i,j=1}^{N} D^\nu(-D_i(a^{ij}_\epsilon D_j u_\epsilon)) = -\sum_{i=1}^{K} D^\nu(b^i_\epsilon u_\epsilon^{n_i}) \text{ in } \Omega \\ D^\nu u_\epsilon|_{\partial\Omega} = D^\nu \rho_\epsilon. \end{aligned} \tag{6.13}$$

Rearranging the above equation and applying the multi-index product rule we find that

$$\begin{aligned} \sum_{i,j=1}^{N} a^{ij}_\epsilon D_{ij}(D^\nu u_\epsilon) = &-\sum_{i,j=1}^{N} D^\nu((D_i a^{ij}_\epsilon)(D_j u_\epsilon)) \\ &-\sum_{i,j=1}^{N} \sum_{\substack{\sigma+\mu=\nu \\ \sigma\neq\nu}} \frac{\nu!}{\sigma!\mu!}(D^\mu a^{ij}_\epsilon)(D^\sigma D_{ij} u_\epsilon) \\ &+\sum_{i=1}^{K} \sum_{\sigma+\mu=\nu} \frac{\nu!}{\sigma!\mu!}(D^\mu b^i_\epsilon)(D^\sigma((u_\epsilon)^{n_i})). \end{aligned} \tag{6.14}$$

Therefore, we may apply Theorem 3.1 to (6.14) to conclude that for an arbitrary multi-index ν such that $|\nu| = k-1$,

$$\begin{aligned}
|D^\nu u_\epsilon|_{2,\gamma;\Omega} \le C\left(\frac{\Lambda_\epsilon}{\lambda_\epsilon}\right)^3 &(|D^\nu u_\epsilon|_{0;\Omega} + |D^\nu \rho_\epsilon|_{2,\gamma;\Omega} \\
&+ |\sum_{i,j=1}^N D^\nu((D_i a_\epsilon^{ij})(D_j u_\epsilon))|_{0,\gamma;\Omega} \\
&+ \sum_{i,j=1}^N \sum_{\substack{\sigma+\mu=\nu \\ \sigma\neq\nu}} \frac{\nu!}{\sigma!\mu!}|D^\mu a_\epsilon^{ij}|_{0,\gamma;\Omega}|D^\sigma D_{ij}u_\epsilon|_{0,\gamma;\Omega} \\
&+ \sum_{i=1}^K \sum_{\sigma+\mu=\nu} \frac{\nu!}{\sigma!\mu!}|D^\mu b_\epsilon^i|_{0,\gamma;\Omega}|D^\sigma((u_\epsilon)^{n_i})|_{0,\gamma;\Omega}) \\
\le C\left(\frac{\Lambda_\epsilon}{\lambda_\epsilon}\right)^3 &(|D^\nu u_\epsilon|_{0;\Omega} + |D^\nu \rho_\epsilon|_{2,\gamma;\Omega} \\
&+ \sum_{i,j=1}^N \sum_{\sigma+\mu=\nu} \frac{\nu!}{\sigma!\mu!}|D^\mu (D_i a_\epsilon^{ij})|_{0,\gamma;\Omega}|D^\sigma (D_j u_\epsilon)|_{0,\gamma;\Omega} \\
&+ \sum_{i,j=1}^N \sum_{\substack{\sigma+\mu=\nu \\ \sigma\neq\nu}} \frac{\nu!}{\sigma!\mu!}|D^\mu a_\epsilon^{ij}|_{0,\gamma;\Omega}|D^\sigma D_{ij}u_\epsilon|_{0,\gamma;\Omega} \\
&+ \sum_{i=1}^K \sum_{\sigma+\mu=\nu} \frac{\nu!}{\sigma!\mu!}|D^\mu b_\epsilon^i|_{0,\gamma;\Omega}|D^\sigma((u_\epsilon)^{n_i})|_{0,\gamma;\Omega}).
\end{aligned} \tag{6.15}$$

By our inductive hypothesis and the assumptions on the coefficients, it is immediate that every term in the above expression is $\mathcal{O}(\epsilon^a)$ for some $a \in \mathbb{R}$ except for the last term. So to show

$$|D^\nu u_\epsilon|_{2,\gamma;\Omega} = \mathcal{O}(\epsilon^a) \quad \text{for some } a \in \mathbb{R},$$

it suffices to show that

$$\sum_{i=1}^K \sum_{\sigma+\mu=\nu} \frac{\nu!}{\sigma!\mu!}|D^\mu b_\epsilon^i|_{0,\gamma;\Omega}|D^\sigma((u_\epsilon)^{n_i})|_{0,\gamma;\Omega} = \mathcal{O}(\epsilon^a) \quad \text{for some } a \in \mathbb{R}.$$

Given that $b_\epsilon^i \in \mathcal{E}_M(\overline{\Omega})$ for each $1 \le i \le K$,

$$|D^\mu b_\epsilon^i|_{0,\gamma;\Omega} = \mathcal{O}(\epsilon^a) \quad \text{for some } a \in \mathbb{R}.$$

Therefore, it is really only necessary to show that for any multi-index σ, such that $|\sigma| = j \le k-1$, that there exists an $a \in \mathbb{R}$ such that

$$|D^\sigma((u_\epsilon)^{n_i})|_{0,\gamma;\Omega} = \mathcal{O}(\epsilon^a).$$

But observe that $D^\sigma((u_\epsilon)^{n_i})$ is a sum of terms of the form

$$(u_\epsilon)^{n_i-m} D^{\sigma_1}u_\epsilon D^{\sigma_2}u_\epsilon \cdots D^{\sigma_m}u_\epsilon,$$

where $\sigma_1+\sigma_2+\cdots\sigma_m=\sigma$ and $m\leq j\leq k-1$. This follows immediately from the chain rule. Therefore we have the following bound:

$$\begin{aligned} |D^\sigma((u_\epsilon)^{n_i})|_{0,\gamma;\Omega} \leq (n_i)|(u_\epsilon)^{n_i-1}|_{0,\gamma;\Omega}|D^\sigma u_\epsilon|_{0,\gamma;\Omega} & \\ + \sum_{\sigma_1+\sigma_2=\sigma}\frac{\sigma!}{\sigma_1!\sigma_2!}(n_i)(n_i-1)|(u_\epsilon)^{n_i-2}|_{0,\gamma;\Omega} & \\ \cdot|D^{\sigma_1}u_\epsilon|_{0,\gamma;\Omega}|D^{\sigma_2}u_\epsilon|_{0,\gamma;\Omega}+\cdots & \\ + \sum_{\sigma_1+\sigma_2+\cdots+\sigma_j=\sigma}\frac{\sigma!}{\sigma_1!\sigma_2!\cdots\sigma_j!}(n_i)(n_i-1) & \\ \cdots(n_i-j)|(u_\epsilon)^{n_i-j}|_{0,\gamma;\Omega}|D^{\sigma_1}u_\epsilon|_{0,\gamma;\Omega} & \\ \cdots|D^{\sigma_j}u_\epsilon|_{0,\gamma;\Omega}. & \end{aligned} \tag{6.16}$$

Using (6.6) and (6.7), for each $m\leq j$ we may bound the terms of the form $|(u_\epsilon)^{n_i-m}|_{0,\gamma;\Omega}$ using $|u_\epsilon|_{0,\gamma;\Omega}$, α'_ϵ and β'_ϵ. Then our inductive hypothesis and the growth conditions on (α'_ϵ) and (β'_ϵ) imply that

$$|D^\sigma((u_\epsilon)^{n_i})|_{0,\gamma;\Omega}=\mathcal{O}(\epsilon^a)\quad\text{for some } a\in\mathbb{R}$$

This implies that

$$|D^\nu u_\epsilon|_{2,\gamma;\Omega}=\mathcal{O}(\epsilon^a)\quad\text{ for some } a\in\mathbb{R}.$$

As ν was an arbitrary multi-index such that $|\nu|=k-1$, this implies there exists $a\in\mathbb{R}$ such that

$$|u_\epsilon|_{k+1,\gamma;\Omega}=\mathcal{O}(\epsilon^a).$$

Therefore, $(u_\epsilon)\in\mathcal{E}_M(\overline{\Omega})$.

Step 5: *Verify that the solution is well-defined.* Proposition 3.8 and the definition of the Dirichlet problem in $\mathcal{G}(\overline{\Omega})$ given in Section 3.5 imply that $[(u_\epsilon)]$ is indeed a solution to the problem

$$\begin{aligned} Au=0 &\ \ \text{in } \Omega,\\ u|_{\partial\Omega}=\rho, & \end{aligned} \tag{6.17}$$

in $\mathcal{G}(\overline{\Omega})$. To see this, we consider other representatives $(\overline{a}^{ij}_\epsilon)$, $(\overline{b}^i_\epsilon)$, $(\overline{\rho}_\epsilon)$, and $(\overline{u}_\epsilon)$ of $[(a^{ij}_\epsilon)]$, $[(b^i_\epsilon)]$, $[(\rho_\epsilon)]$, and $[(u_\epsilon)]$. Then the proof of Proposition 3.8 clearly implies that

$$\begin{aligned} -\sum_{i,j=1}^N D_i(\overline{a}^{ij}_\epsilon D_j\overline{u}_\epsilon)+\sum_{i=1}^K\overline{b}^i_\epsilon(\overline{u}_\epsilon)^{n_i}=\eta_\epsilon &\ \ \text{in } \Omega,\\ \overline{u_\epsilon}|_{\partial\Omega}=\overline{\rho}_\epsilon+\overline{\eta}_\epsilon, & \end{aligned} \tag{6.18}$$

where $\eta_\epsilon\in\mathcal{N}(\overline{\Omega})$ and $\overline{\eta}_\epsilon$ is a net of functions satisfying (3.16). But this implies that this choice of representatives also satisfies (6.17) in $\mathcal{G}(\overline{\Omega})$, so our solution $[(u_\epsilon)]$ is independent of the representatives used.

□

This completes our proof of Theorem 4.1. We now conclude this article by giving a brief summary of everything that we have discussed.

7. SUMMARY

We began the paper with an example to motivate the Colombeau Algebra method of solving the semilinear problem (1.1) with potentially distributional data. In particular, in Section 2 we proved the existence of a solution to an ill-posed critical exponent problem in Proposition 2.4. Our method of proving the existence of a solution to this problem consisted of mollifying the data of the original problem and solving a sequence of ”approximate” problems with the smooth coefficients. We then obtained a sequence of solutions that yielded a convergent subsequence. The framework we used to obtain a solution to this problem was modeled on the more general Colombeau approach that we developed later in the paper, but required only basic elliptic PDE theory. Following this existence proof, we began to develop the more general Colombeau algebra framework. To this end, in Section 3.1 we introduced notation for Holder norms and stated two *a priori* estimates from [5] that were made more precise by Mitrovic and Pilipovic in [12]. In Section 3.2 we then introduced the general framework for constructing Colombeau-type algebras and the Colombeau algebra $\mathcal{G}(\overline{\Omega})$ used in this paper. We then discussed a method used to embed the Schwartz distributions $\mathcal{D}'(\Omega)$ into $\mathcal{G}(\overline{\Omega})$. We used this embedding to analyze a problem of the form (1.1) with distributional coefficients. In particular, in Proposition 4.5 we determined explicit conditions under which we could solve a semilinear problem of the form (1.1) with rough coefficients. Then we finished Section 3.2 by defining a class of semilinear operators on $\mathcal{G}(\overline{\Omega})$ in 3.4, and we then defined the Dirichlet problem for these operators.

Our main results for the Colombeau algebra framework were presented in Section 4, namely Theorem 4.1, which consists of an existence result for the semilinear problem (4.2) in $\mathcal{G}(\overline{\Omega})$. We then developed the necessary tools to analyze our semilinear problem in Section 5. We first determined a net of L^∞ bounds for positive solutions to our problem. Then, in Section 5.2 we showed that this net of L^∞ bounds is in fact a net of sub- and super-solutions contained in $\overline{\mathbb{C}}$, the ring of generalized constants described in Section 3.2. After developing our sub- and super-solutions, we proved Theorem 4.1 in Section 6. We set up our problem in a manner similar to that used by Mitrovic and Pilipovic in [12]. However, our approach to solving our semilinear problem was distinct from theirs; we first determined a net of solutions (u_ϵ) to the family of semilinear problems (6.3) by using the method of sub-and super-solutions (Theorem 4.3), and our net of sub- and super-solutions determined in Section 5.2. Once our net of solutions was determined, we then employed Theorems 3.1 and our net of sub- and super-solutions to show that our net of solutions was contained in $\mathcal{E}_M(\overline{\Omega})$.

In this article we have attempted to develop some basic tools to allow for a more general study of the Einstein constraint equations with distributional data. Our goal was to extend the current solution theory for scalar, critical exponent semilinear problems such as the Lichnerowicz equation, allowing for more irregular data than is currently covered by the existing solutions theories (cf. [7, 8] for a summary of the known results for the CMC, near-CMC, and Far-CMC cases through 2009). As a next step, we hope to use the tools developed in this article to extend the near-CMC and Far-CMC existence framework for rough metrics developed in [8, 9, 10, 2] to cover the rough data example studied by Maxwell in [11].

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